# Math 8250: Day 17

**Def:** An *n-manifold w/ boundary* $M^n$ is a manifold parametrized by coord. charts $\{(U_\alpha, \varphi_\alpha)\}$ where the $U_\alpha$ are open subsets of the upper half space $H^k = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The *boundary* of $M$, denoted $\partial M$, consists of those points $p \in M$ s.t. $\varphi_\alpha^{-1}(p) = (x_1, \dots, x_{n-1}, 0)$ for all $\alpha$ s.t. $p \in \varphi_\alpha(U_\alpha)$. The *interior* of $M$ is $\mathrm{int}(M) = M - \partial M$.

**Ex:** $B^n$, $\mathbb{R}^n - B^n$, $S^n - B_\varepsilon(N)$, etc.

**Stokes' Theorem:** If $M^n$ is a smooth, oriented, $n$-dimensional manifold (possibly w/ boundary), and $\omega \in \Omega^{n-1}(M)$ is compactly supported, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

**Cor:** If $M$ is closed, then $\int_M d\omega = 0$.

**Proof:** Since both $\int_M d\omega$ & $\int_{\partial M} \omega$ are linear in $\omega$, it suffices to prove this when the (compact) support of $\omega$ is contained in a single coord. chart $\varphi: U \to M$, where $U \subset H^n$ is open.

If $U \subset \mathrm{int}(H^k)$, then, since $\mathrm{supp}(\omega) \subset \varphi(U)$ and $\varphi(U) \cap \partial M = \emptyset$, we have

$$\int_{\partial M} \omega = 0.$$

Now,

$$\int_M d\omega = \int_{\varphi(U)} d\omega = \int_U \varphi^* d\omega = \int_U d\varphi^* \omega.$$

Since $\varphi^*\omega$ is an $(n-1)$-form on $\mathbb{R}^n$, we can write

$$\varphi^*\omega = \sum_{k=1}^{n} (-1)^{k-1} f_k \, dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n$$

so that

$$d(\varphi^*\omega) = \sum_{k=1}^{n} (-1)^{k-1} \left(\frac{\partial f_k}{\partial x_k} dx_k\right) \wedge dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n = \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_k} dx_1 \wedge \dots \wedge dx_n.$$

But then

$$\int_U d\varphi^*\omega = \sum_{k=1}^{n} \int_{\mathbb{R}^n} \frac{\partial f_k}{\partial x_k}\, dx_1 \cdots dx_n .$$

Now use Fubini's theorem to integrate the $i^{th}$ term first:

$$\int_{\mathbb{R}^n} \frac{\partial f_k}{\partial x_k}\, dx_1 \cdots dx_n = \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} \frac{\partial f_k}{\partial x_k}\, dx_k \right) dx_1 \cdots \widehat{dx_k} \cdots dx_n .$$

But, by the FTC, $\displaystyle\int_{-\infty}^{\infty} \frac{\partial f_k}{\partial x_k}(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)\, dt = \lim_{a \to \infty} \left[ f_k(x_1, \dots, x_{k-1}, a, x_{k+1}, \dots, x_n) - f_k(x_1, \dots, x_{k-1}, -a, x_{k+1}, \dots, x_n) \right]$

$$= 0$$

since $\varphi^*\omega$ (and hence $f_k$) is compactly-supported.

Thus, we conclude that $\int_{\partial M} \omega = 0$. Since we already know $\int_M d\omega = 0$, Stokes' thm holds in this case.

Now, when $U \cap \partial(H^n) \neq \emptyset$, all of the above goes through except when $k=n$. Since $\partial(H^n) = \{x_n = 0\}$, the last integral is different & we have

$$\int_U d\varphi^*\omega = \int_{H^n} \frac{\partial f_n}{\partial x_n}\, dx_1 \cdots dx_n = \int_{\mathbb{R}^{n-1}} \left( \int_0^{\infty} \frac{\partial f_n}{\partial x_n}\, dx_n \right) dx_1 \cdots dx_{n-1} .$$

Now $\int_0^{\infty} \frac{\partial f_n}{\partial x_n}\, dx_n = 0 - f_n(x_1, \dots, x_{n-1}, 0)$, so

$$\int_M d\omega = \int_U d\varphi^*\omega = \int_{\mathbb{R}^{n-1}} -f_n(x_1, \dots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1} .$$

On the other hand,

$$\int_{\partial M} \omega = \int_{\partial U} \varphi^*\omega = \int_{\partial H^n} \varphi^*\omega = \sum_{k=1}^{n} \int_{\partial H^n} (-1)^{k-1} f_k\, dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n .$$

But $x_n = 0$ on $\partial H^n$, so $dx_n = 0$ on $\partial H^n$, so the only term above that lives is

$$\int_{\partial H^n} (-1)^{n-1} f_n(x_1, \dots, x_{n-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{n-1}$$

Now, I want to identify $\partial \mathbb{H}^n$ with $\mathbb{R}^{n-1}$ via the inclusion

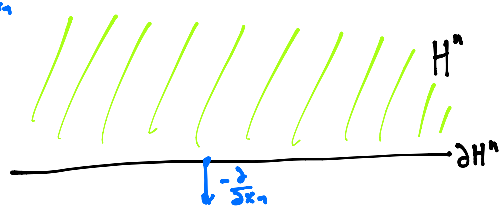
$$i : \mathbb{R}^{n-1} \to \partial \mathbb{H}^n$$
$$(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 0)$$

but I have to be very careful about orientations, which means I need to say what the induced orientation on the boundary of a manifold is.

**Def:** If $M^n$ is an oriented manifold w/ nonempty boundary $\partial M \neq \emptyset$, then the **boundary orientation** on the boundary $\partial M$ is defined as follows: for $p \in \partial M$, let $(U, \varphi)$ be a coord. chart centered at $p$ (i.e. $U \subset \mathbb{H}^n$ open & $\varphi(\vec{0}) = p$). Then $v_p := d\varphi_{\vec{0}}(-\vec{e}_n)$ is the **unit outward normal vector** at $p$ and we choose the orientation on $T_p\partial M$ by requiring that, if $(v_1, \dots, v_{n-1})$ agrees w/ the orientation on $T_p \partial M$, then $(v_p, v_1, \dots, v_{n-1})$ agrees w/ the given orientation on $T_pM$. Equivalently, if $\omega \in \Omega^n(M)$ is a nonvanishing $n$-form on $M$ that agrees w/ the orientation, then $\iota_v \omega$ defines the boundary orientation on $\partial M$.

**Ex:** Consider $\mathbb{H}^n$ w/ the std. orientation $dx_1 \wedge \dots \wedge dx_n$. Then the unit outward normal on $\partial \mathbb{H}^n$ is $-\frac{\partial}{\partial x_n}$.

Then $\iota_{-\frac{\partial}{\partial x_n}} dx_1 \wedge \dots \wedge dx_n = (-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$ gives the orientation on $\partial \mathbb{H}^n$.




Returning to the proof of Stokes' Theorem, we see that

$$\int_{\partial M} \omega = \int_{\partial \mathbb{H}^n} (-1)^{n-1} f_n \, dx_1 \wedge \dots \wedge dx_{n-1} = (-1)^n \int_{\mathbb{R}^{n-1}} (-1)^{n-1} i^*(f_n \, dx_1 \wedge \dots \wedge dx_{n-1}) = -\int_{\mathbb{R}^{n-1}} f_n \, dx_1 \wedge \dots \wedge dx_{n-1}$$

But we already showed that the RHS is equal to $\int_M d\omega$, so we're done! $\blacksquare$