Math 8250: Day 16

Now that we have a definition of integration, it's good to check that it agrees with versions of integration that we already know. The first key tool is:

Thm: If $f: M^n \to N^n$ is an orientation-preserving diffeo., then

$$\int_N \omega = \int_M f^* \omega$$

for all compactly-supported $\omega \in \Omega^n(N)$.

Proof is just choosing local coords, since the result for $M, N \subset \mathbb{R}^n$ was the cornerstone of our whole theory of integration on manifolds.

In fact, it turns out that the integral we've defined is the unique linear operator on compactly-supported forms that satisfies the above theorem & reduces to the usual integration on Euclidean space.

Examples:

① Suppose $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \in \Omega^1(\mathbb{R}^3)$ and $\gamma: [0,1] \to \mathbb{R}^3$ is an embedding. Then of course $\gamma: [0,1] \to C = \gamma([0,1])$ is an orientation preserving diffeo. Hence

$$\int_C \omega = \int_{[0,1]} \gamma^* \omega.$$

Now, if $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, then

$$\gamma^* dx_i = d\gamma_i = \frac{d\gamma_i}{dt} dt,$$

so

$$\int_C \omega = \int_{[0,1]} \gamma^* \omega = \int_{[0,1]} \left(f_1(\gamma(t)) \frac{d\gamma_1}{dt} + f_2(\gamma(t)) \frac{d\gamma_2}{dt} + f_3(\gamma(t)) \frac{d\gamma_3}{dt} \right) dt = \sum_{i=1}^{3} \int_0^1 f_i(\gamma(t)) \frac{d\gamma_i}{dt} dt$$

Now this should look familiar. If $\vec{F} = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3} = \omega^\#$, then the RHS is $\oint \vec{F} \cdot d\gamma$, the line integral of $\vec{F}$ over C.

(2) Let $\eta = *\omega = f_1\,dx_2\wedge dx_3 + f_2\,dx_3\wedge dx_1 + f_3\,dx_1\wedge dx_2 \in \Omega^2(\mathbb{R}^3)$

Let S be a sfce which is the graph of some fctn $G:\mathbb{R}^2\to\mathbb{R}$; i.e.,

$$S=\{(x_1,x_2,G(x_1,x_2))\mid x_1,x_2\in\mathbb{R}\}.$$

Now, if $h:\mathbb{R}^2\to S$ is the obvious parametrization $h(x_1,x_2)=(x_1,x_2,G(x_1,x_2))$, then

$$\int_S \omega = \int_{\mathbb{R}^2} h^*\omega$$

provided these integrals exist (e.g. if ω is compactly supported)

But now

$$h^*(dx_1\wedge dx_2) = dx_1\wedge dx_2$$

$$h^*(dx_2\wedge dx_3) = dx_2\wedge dG = dx_2\wedge\left(\frac{\partial G}{\partial x_1}dx_1+\frac{\partial G}{\partial x_2}dx_2\right) = -\frac{\partial G}{\partial x_1}dx_1\wedge dx_2$$

$$h^*(dx_3\wedge dx_1) = dG\wedge dx_1 = \left(\frac{\partial G}{\partial x_1}dx_1+\frac{\partial G}{\partial x_2}dx_2\right)\wedge dx_1 = -\frac{\partial G}{\partial x_2}dx_1\wedge dx_2.$$

Therefore,

$$\int_S\omega = \int_{\mathbb{R}^2}\left(-f_1\frac{\partial G}{\partial x_1} - f_2\frac{\partial G}{\partial x_2} + f_3\right)dx_1\wedge dx_2$$

Now, if $x=(x_1,x_2,G(x_1,x_2))\in S$, then

$$\vec{n}(x) = \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1\right)$$

is normal to S at x. Then if we define $\vec{u}=\frac{\vec{n}}{|\vec{n}|}$ and let $\vec{F}=(*\eta)^\#=f_1\frac{\partial}{\partial x_1}+f_2\frac{\partial}{\partial x_2}+f_3\frac{\partial}{\partial x_3}$ and

$dA = |\vec{n}|\,dx_1\wedge dx_2$, then

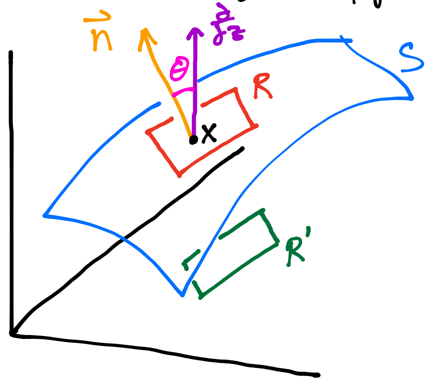
$$\int_S\omega = \int_{\mathbb{R}^2}\left(-f_1\frac{\partial G}{\partial x_1} - f_2\frac{\partial G}{\partial x_2} + f_3\right)dx_1\wedge dx_2$$

$$= \int_{\mathbb{R}^2}(\vec{F}\cdot\vec{u})\,dA,$$

which should also be a familiar expression.

What is this weirdo dA? Well, it's usually called the area form of the surface S for the following reason:

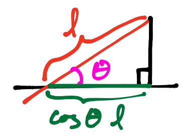
If R is a small rectangle in $\mathbb{R}^3$ tangent to S at x and R' is its projection to the x_1x_2-plane



Then

$$\text{Area}(R') = \cos\theta \, \text{Area}(R)$$

Why? Well if I translate R until it intersects the x_1x_2-plane, then one direction lies in the plane & one direction is at an angle θ w/ the plane. In the projection, the first direction is scaled not at all & the second by $\cos\theta$:



Now, at $x = (x_1, x_2, G(x_1,x_2)) \in S$, we have that

$$\cos\theta = \left\langle \vec{u}, \frac{\partial}{\partial x_3} \right\rangle = \left\langle \frac{\vec{n}}{|\vec{n}|}, \frac{\partial}{\partial x_3} \right\rangle = \frac{1}{|\vec{n}|} \left\langle \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1\right), (0,0,1) \right\rangle = \frac{1}{|\vec{n}|},$$

So

$$\text{Area}(R) = \frac{1}{\cos\theta} \text{Area}(R') = |\vec{n}| \, \text{Area}(R') = |\vec{n}| \int_{R'} dx_1 \wedge dx_2 = \int_{R'} dA$$

Now, I claimed at the beginning of the semester that differential forms give a way to synthesize the Fundamental Theorem of Calculus, Green's Theorem, & the Divergence theorem into a single statement.

Let's see what these theorems now say in this language:

FTC: If $f(x) = F'(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Okay, well $f(x)\,dx = dF$, and $F(b) - F(a) = \int_{\partial[a,b]} F(x)\,dx$,

So FTC just says

$$\int_{[a,b]} dF = \int_{\partial([a,b])} F$$

Green's Thm: If $\vec{F}$ is a v.f. on $\mathbb{R}^3$, then

$$\int_S (\nabla \times \vec{F}) \cdot dS = \oint \vec{F} \cdot d\gamma \qquad \text{where } C = \gamma([0,1]) = \partial S$$

But now, if $\omega = \vec{F}^\flat$, we have that $\int_S (\nabla \times \vec{F}) \cdot dS = \int_S d\omega$

and $\oint \vec{F} \cdot d\gamma = \int_C \omega$

So Green's Thm says $\int_S d\omega = \int_{\partial S} \omega$ for surfaces in $\mathbb{R}^3$.

At this point, hopefully the pattern is clear. Then the synthesis is:

Stokes' theorem: If D is a regular domain in an oriented n-mfd M (including $D = M$) and $\omega \in \Omega^{n-1}(M)$

has compact support, then

$$\int_D d\omega = \int_{\partial D} \omega.$$

Proof to come.