Math 8250: Day 15

More on orientations:

Recall from HW 2 #6 that an orientation of an n-dimensional vector space V is a choice of component of

$$\Lambda^n(V) - \{0\} \cong \mathbb{R} - \{0\}$$

Alternatively, we could think of an orientation of V as a choice of element of the following (setwise) quotient:

$$\{(v_1, \dots, v_n) \mid v_1, \dots, v_n \text{ a basis for } V\} / \sim$$

where $(v_1, \dots, v_n) \sim (w_1, \dots, w_n)$ if the change-of-basis matrix $A: v_i \mapsto w_i$ has positive determinant.

Then a representative of an orientation is just a choice of basis.

Why is this equiv. to the other def.?

Well, suppose $v_1, \dots, v_n$ & $w_1, \dots, w_n$ are two different bases for V.

Then $v_1 \wedge \dots \wedge v_n,\ w_1 \wedge \dots \wedge w_n \in \Lambda^n(V) - \{0\}$.

Clearly, $\quad w_1 \wedge \dots \wedge w_n = (Av_1) \wedge \dots \wedge (Av_n) = A(v_1 \wedge \dots \wedge v_n) = (\det A)\, v_1 \wedge \dots \wedge v_n$

by HW2#2(c). Therefore, $v_1 \wedge \dots \wedge v_n$ & $w_1 \wedge \dots \wedge w_n$ are in the same component of $\Lambda^n(V) - \{0\}$

if & only if $v_1, \dots, v_n$ & $w_1, \dots, w_n$ determine the same orientation of V.

Now, we could also define an orientation on V to be a choice of component of $\Lambda^n(V^*) - \{0\}$.

Why? Well, suppose $\omega \in \Lambda^n(V^*) - \{0\}$ and $v_1, \dots, v_n$ & $w_1, \dots, w_n$ are bases for V.

Then $\omega(v_1, \dots, v_n) = a$ & $\omega(w_1, \dots, w_n) = b$ for some non-zero $a, b \in \mathbb{R}$

But then $b = \omega(w_1, \dots, w_n) = \omega(Av_1, \dots, Av_n) = A^*\omega(v_1, \dots, v_n) = (\det A^*)\, \omega(v_1, \dots, v_n) = (\det A) \cdot a$.

So a & b have the same sign $\iff \det A>0 \iff v_1,\dots,v_n$ & $w_1,\dots,w_n$ represent the same orientation on V

Clearly, scaling $\omega\in\Lambda^n(V^*)-\{0\}$ doesn't chg the sign of $\omega(v_1,\dots,v_n)$, so the orientation on V determines a component of $\Lambda^n(V^*)-\{0\}$, and conversely a component of $\Lambda^n(V^*)-\{0\}$ determines an orientation on V (namely those bases $v_1,\dots,v_n$ so that $\omega(v_1,\dots,v_n)>0$ for any ω in the chosen component).

Even more specifically, if $v_1,\dots,v_n$ is a basis for V and $v_1^*,\dots,v_n^*$ is the corresponding dual basis for V^*, then any elt of $\Lambda^n(V^*)-\{0\}$ can be written as $a\, v_1^*\wedge\dots\wedge v_n^*$ for some $a\neq 0$. Those elts w/ $a>0$ exactly comprise the preferred component.

Now, w/ all this machinery at our disposal, we can give a very concise definition of the orientation of a mfld:

Def: Let M^n be connected. Then the zero section of the exterior n-bundle $\Lambda^n(M)$ is

$$O:=\bigcup_{p\in M}\{0\in\Lambda^n((T_pM)^*)\}$$

Since each $\Lambda^n((T_pM)^*)-\{0\}$ has 2 components, $\Lambda^n(M)-O$ has at most 2 components. M is orientable if $\Lambda^n(M)-O$ has exactly 2 components, and an orientation on M is a choice of one of the components.

What could go wrong?

Well, $\Lambda^n(M)-\{0\}$ could have only one component. Schematically, since $\Lambda^n((T_pM)^*)$ is 1-dim'l, the picture would be something like... (prop Möbius strip)

Since picking a component of $\Lambda^n(V^*)-\{0\}$ is equivalent to picking an equiv. class of bases of V, this could fail if there's no globally consistent way of picking bases for T_pM. In other words:

Prop: TFAE:

① M is orientable in the above sense

② M is orientable in the sense of HW 1 #1

③ $\exists$ a nonvanishing n-form on M.

Pf: ③ ⇒ ① Clearly a nonvanishing $\omega \in \Omega^n(M)$ determines disjoint

$$\Lambda_+ = \bigcup_{p \in M} \{a\omega_p : a > 0\}$$

$$\Lambda_- = \bigcup_{p \in M} \{a\omega_p : a < 0\}$$

so $\Lambda^n(M) - 0 = \Lambda_+ \cup \Lambda_-$ is disconnected

① ⇒ ② M orientable, so choose an orientation Λ, which is a component of $\Lambda^n(M) - 0$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be any collection of coord. charts & let $\{(V_\beta, \psi_\beta)\}$ be the subcollection s.t., if $x_1, \ldots, x_n$ are the associated local coords. at $p \in M$, then $(dx_1 \wedge \cdots \wedge dx_n)_p \in \Lambda$ for all $p \in M$.

Then if (V_1, ψ_1) & (V_2, ψ_2) are 2 charts in a nbhd of $q \in M$, with associated coords $x_1, \ldots, x_n$ & $y_1, \ldots, y_n$ then we have that $(dx_1 \wedge \cdots \wedge dx_n)_q = (\psi_2^{-1} \circ \psi_1)^* (dy_1 \wedge \cdots \wedge dy_n)_q = \det(\psi_2^{-1} \circ \psi_1)^* (dy_1 \wedge \cdots \wedge dy_n)_q$.

Since $(dx_1 \wedge \cdots \wedge dx_n)_q$ & $(dy_1 \wedge \cdots \wedge dy_n)_q$ belong to the same component of $\Lambda^n((T_q M)^*) - \{0\}$, it must be the case that $\det(\psi_2^{-1} \circ \psi_1)^* = \det(d(\psi_2^{-1} \circ \psi_1)_q) > 0$. Since this is true for all q lying in overlapping coord. charts, ② follows.

② ⇒ ③ Suppose $\{(U_\alpha, \varphi_\alpha)\}$ is a collection of coord. charts so that $\det(d(\varphi_\beta^{-1} \circ \varphi_\alpha))$ is positive on all overlaps. Let $\{(U_i, \varphi_i)\}$ be a countable (including finite) subsystem of charts w/ associated local coords $x_1^i, \ldots, x_n^i$. Let $\{\rho_i\}$ be a subordinate partition of unity. Then define the form

$$\omega := \sum_i \rho_i (dx_1^i \wedge \cdots \wedge dx_n^i)$$

Then each $(dx_1^i \wedge \cdots \wedge dx_n^i)_p$ that is defined lies in the same component of $\Lambda^n((T_p M)^*)$, so ω_p never vanishes. ∎