Math 8250: Day 14

Integration on Manifolds

In order to define integration on manifolds, let's first think about integration on Euclidean space:

Let $A \subset \mathbb{R}^n$ be a bounded open set. Let $f: A \to \mathbb{R}$ be a smooth function.

Then

$$\int_A f = \int_A f \, dy_1 \cdots dy_n$$

is defined by, e.g., Riemann sums.

Now, our goal is to integrate on mfds, where we don't have canonical coords... so what happens when we change coords?

Well, if $\varphi: A \to B \subset \mathbb{R}^n$ is a diffeomorphism, then

$$\int_B f \, dx_1 \cdots dx_n = \int_A f \circ \varphi \; \det(d\varphi) \, dy_1 \cdots dy_n$$

where I use $x_1, \dots, x_n$ as coords. on B to emphasize that it's a diff't space.

Of course, this is weird & unnatural if you think about it. What are $dx_1 \cdots dx_n$ & $dy_1 \cdots dy_n$, really? What's w/ the $\det(d\varphi)$ term? Specifically, what is it on a mfd?

Here's the strategy: rather than integrate the function f, we will integrate the n-form $\omega = f \, dx_1 \wedge \cdots \wedge dx_n$.

We define

$$\int_B \omega := \int_B f \, dx_1 \cdots dx_n$$

where the RHS is the usual Riemann integral. Since every n-form on $\mathbb{R}^n$ is of the form $\eta = g \, dx_1 \wedge \cdots \wedge dx_n$ this definition is unambiguous.

Now the change of variables is easy to state (recall $\varphi: A \to B$, so $\varphi^*: \Omega^n(B) \to \Omega^n(A)$):

$$\boxed{\int_B \omega = \int_A \varphi^*\omega} = \int_A \varphi^*(f\, dx_1\wedge\cdots\wedge dx_n) = \int_A (f\circ\varphi)\,\varphi^*(dx_1\wedge\cdots\wedge dx_n)$$

$$= \int_A (f\circ\varphi)\,\det(d\varphi)\, dy_1\wedge\cdots\wedge dy_n$$

using the fact that, if $F: V\to V$ is linear and V is n-dimensional, then the induced map $\Lambda^n(V)\to\Lambda^n(V)$ is multiplication by $\det F$ (this was HW 2 #2(c)) plus the fact that $\det(\varphi^*) = \det(d\varphi)$ since by definition φ^* is the transpose of $d\varphi$.

The stuff in the red box is now completely notated in the language of forms, yet recovers the usual change of coords. formula.

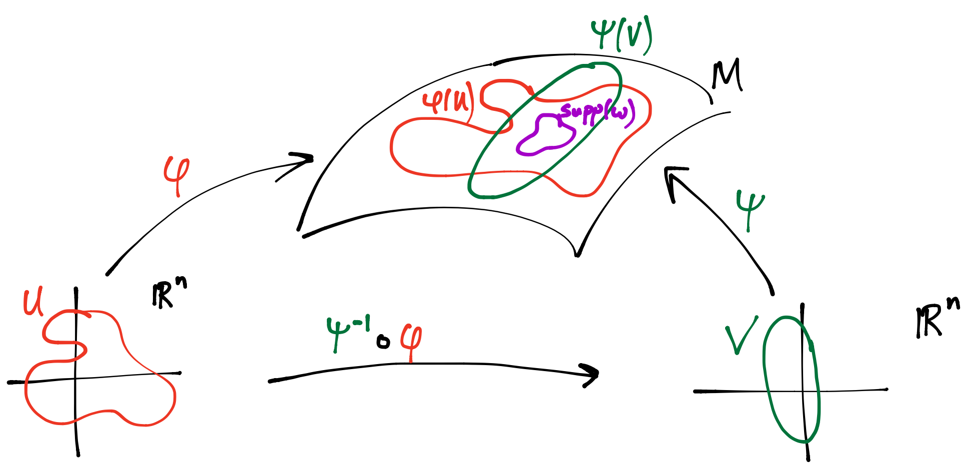
This, then, will allow us to define integration on manifolds... almost. We will need our mfds to be oriented,

Here's how it works:

If $\omega\in\Omega^n(M)$ and $\mathrm{supp}(\omega)$ is contained in the image $\varphi(U)$ of a single coord. chart $\varphi: U\subset\mathbb{R}^n\to M$, then we can define

$$\int_M \omega := \int_U \varphi^*\omega.$$

What if $\mathrm{supp}(\omega)$ is also contained in the image of a different coord. chart $\psi: V\subset\mathbb{R}^n\to M$?



Then $\theta := \psi^{-1}\circ\varphi : \varphi^{-1}(\mathrm{supp}(\omega)) \to \psi^{-1}(\mathrm{supp}(\omega))$ is an (orientation-preserving) diffeo., so

$$\int_V \psi^*\omega = \int_U \theta^*\psi^*\omega = \int_U (\psi\circ\theta)^*\omega = \int_U (\psi\circ\psi^{-1}\circ\varphi)^*\omega = \int_U \varphi^*\omega = \int_M \omega,$$

so this is well-defined.

Of course, it generally won't be the case that the support of an n-form is contained in a single coord. chart, but we still want to integrate on $\mathbb{R}^n$.

Q: How can we achieve this?

A: With a partition of unity!

Specifically, suppose M^n is a smooth, oriented mfd & $\omega \in \Omega^n(M)$ is compactly supported.

If $\{(U_i, \varphi_i)\}$ is a countable, oriented collection of coordinate charts & $\{\rho_i\}$ is a subordinate partition of unity, then define

$$\int_M \omega := \sum_i \int_M \rho_i\,\omega.$$

Since ω is compactly-supported, the finiteness property of the partition of unity $\{\rho_i\}$ ensures that only finitely many ρ_i are nonzero on $\mathrm{supp}(\omega)$, so the RHS is a finite sum & we don't have to worry about convergence.

Each $\rho_i\omega$ is a compactly-supported n-form w/ $\mathrm{supp}(\rho_i\omega) \subset \varphi_i(U_i)$, so $\int_M \rho_i\omega$ is unambiguously defined by our work above.

Therefore, to see that $\int_M \omega$ is well-defined we need only show that it is independent of the partition of unity.

Suppose $\{\rho_j'\}$ is a different partition of unity. Then, since $\sum_j \rho_j' = 1$ at each $x \in M$, we know

$$\sum_j \rho_j' \rho_i \omega_p = \rho_i \omega.$$

Hence,

$$\int_M \rho_i \omega = \int_{U_i} \varphi_i^* \rho_i \omega = \int_{U_i} \varphi_i^* \Big(\sum_j \rho_j' \rho_i \omega\Big) = \int_{U_i} \sum_j \varphi_i^* \rho_j' \rho_i \omega = \sum_j \int_{U_i} \varphi_i^* \rho_j' \rho_i \omega = \sum_j \int_M \rho_j' \rho_i \omega$$

by the linearity of the pullback, the linearity of integration on $\mathbb{R}^n$, and the fact that $\operatorname{supp}(\rho_j' \rho_i \omega) \subset \operatorname{supp}(\rho_i \omega) \subset \varphi_i(U_i)$ for all j.

Likewise $\int_M \rho_j' \omega = \sum_i \int_M \rho_i \rho_j' \omega$, so we see that

$$\sum_i \int_M \rho_i \omega = \sum_i \sum_j \int_M \rho_j' \rho_i \omega = \sum_j \sum_i \int_M \rho_i \rho_j' \omega = \sum_j \int_M \rho_j' \omega,$$

so $\int_M \omega$ is the same when calculated w.r.t. either partition.