Math 8250: Day 13

Cartan's Magic Formula: If $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$, then

$$L_X \omega = \iota_X d\omega + d\iota_X \omega.$$

Proof: If $\omega = u \in \Omega^0(M)$, then we know $L_X u = X(u)$. Also,

$$\iota_X du + d(\iota_X u) = du(X) + 0 = X(u),$$

So this holds for 0-forms.

Moreover, for any $Y \in \mathfrak{X}(M)$ and $p \in M$,

$$(L_X du)_p Y = \frac{d}{dt}\Big|_{t=0} \varphi_t^*(du_{\varphi_t(p)})\, Y = \frac{d}{dt}\Big|_{t=0} du(d\varphi_t Y) = \frac{d}{dt}\Big|_{t=0} d(u\circ\varphi_t)(Y) = \frac{d}{dt}\Big|_{t=0} Y(u\circ\varphi_t)$$

$$= Y\left(\frac{d(u\circ\varphi_t)}{dt}\Big|_{t=0}\right)$$

$$= Y(X(u))$$

But then

$$\iota_X d(du) = 0$$

and

$$d(\iota_X du)(Y) = d(du(X))(Y) = d(X(u))(Y) = Y(X(u)).$$

So the theorem holds for differentials.

Now, if $\omega = \sum a_I\, dx_I$, then

$$L_X \omega = \sum L_X a_I\, dx_I + a_I L_X dx_I = \sum (\iota_X d + d\iota_X) a_I\, dx_I + a_I (\iota_X d + d\iota_X)\, dx_I = (\iota_X d + d\iota_X) \sum a_I\, dx_I$$

$$= (\iota_X d + d\iota_X)\, \omega. \quad \blacksquare$$