Math 8250: Day 12

Using HW problem 2(b), if $f: M^m \to N^n$ is smooth, then we get a map $f^*: \Omega^*(N) \to \Omega^*(M)$ given by

$$f^*\omega_p(X_1, \ldots, X_k) = \omega_{f(p)}(df_p X_1, \ldots, df_p X_k)$$

Prop: $d(f^*\omega) = f^* d\omega$ for any $\omega \in \Omega^k(N)$.

Proof: Let $p \in M$ and let $x_1, \ldots, x_n$ be local coords. in a nbhd of $f(p)$.

First, note that if $\omega = u \in \Omega^0(N)$, then

$$f^*u = u \circ f$$

so

$$df^*u = d(u \circ f).$$

OTOH,

$$f^*du(X) = du(df_p X) = (df_p X)(u) = X(u \circ f) = d(u \circ f)(X) \quad \text{for all } X \in T_pM,$$

so

$$f^*du = d(u \circ f)$$

Now, if $\omega \in \Omega^k(N)$, then in local coords

$$\omega_{f(p)} = \sum a_I \, dx_I,$$

so

$$(d\omega)_{f(p)} = \sum_I da_I \wedge dx_I$$

And so

$$(f^*d\omega)_p \overset{HW}{=} \sum_I f^*(da_I) \wedge f^*(dx_I) = \sum_I d(a_I \circ f) \wedge d(x_I \circ f)$$

OTOH, $(f^*\omega)_p = \sum_I (a_I \circ f) \, d(x_I \circ f)$

so $(df^*\omega)_p = \sum_I d(a_I \circ f) \wedge d(x_I \circ f)$

Lie derivatives of forms (& tensors)

Recall the following proposition:

Prop: Let X and Y be smooth vector fields on M, let $p \in M$, and let φ_t be the local flow of X in a nbhd U of p. Then

$$[X,Y](p) = \lim_{t\to 0} \frac{1}{t}\left[Y_{\varphi_t(p)} - d\varphi_t(Y_p)\right] \quad \text{or equivalently} \quad \lim_{t\to 0} \frac{1}{t}\left[d\varphi_{-t}\left(Y_{\varphi_t(p)}\right) - Y_p\right]$$

The expression on the RHS is also called the **Lie derivative** of Y w.r.t. X, denoted $L_X Y$.

Prop: $L_{[u,v]} = [L_u, L_v]$ as operators on $\mathfrak{X}(M)$.

Proof: Let $W \in \mathfrak{X}(M)$. Then

$$L_{[u,v]} W = [[u,v],W]$$

$$L_u L_v W = [u,[v,W]] = -[[v,W],u]$$

$$L_v L_u W = [v,[u,W]] = -[[u,W],v] = [[W,u],v]$$

So, by the Jacobi identity,

$$L_{[u,v]} W - L_u L_v W + L_v L_u W = 0$$

or, equivalently,

$$L_{[u,v]} W = L_u L_v W - L_v L_u W = [L_u, L_v] W$$

$\blacksquare$

Analogously, we can define the **Lie derivative** of a form ω w.r.t. a vector field X as

$$(L_X \omega)_p := \lim_{t\to 0} \frac{1}{t}\left[\varphi_t^*(\omega_{\varphi_t(p)}) - \omega_p\right] = \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_t^*(\omega_{\varphi_t(p)})\right)$$

In fact, this can be extended to arbitrary tensors in the obvious way: for $\alpha \in T^0_r(M)$ & $\beta \in T^s_0(M)$,

$$(L_X \alpha \otimes \beta)_p = \left.\frac{d}{dt}\right|_{t=0} \left(d\varphi_t(\alpha_{\varphi_t(p)}) \otimes \varphi_t^*(\beta_{\varphi_t(p)})\right)$$

Prop: For $f \in C^\infty(M) = \Omega^0(M)$ and $X \in \mathfrak{X}(M)$,

$$L_X f = X(f)$$

Proof: This is essentially the definition.

Cartan's Magic Formula: If $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$, then

$$L_X \omega = \iota_X d\omega + d\iota_X \omega.$$

Proof to come, but what does this get you?

Ex: Consider S^3 as the set of unit quaternions. Then at each point p on the sphere, a tangent vector $v \in T_pS^3$ is just a quaternion which is perpendicular to the quaternion p.

For example, at $1 \in S^3$, the tangent space consists of all purely imaginary quaternions $ai+bj+ck$.

Now, I will define 3 mutually perpendicular vector fields X, Y, Z on S^3 by

$$X_p = pi, \quad Y_p = pj, \quad Z_p = pk$$

Check: $\langle p, pi\rangle = \langle p_0+p_1i+p_2j+p_3k, -p_1+p_0i+p_3j-p_2k\rangle = 0$ ✓

and likewise for the others, so these really are tangent vectors.

Similarly, $\langle pi, pj\rangle = \langle -p_1+p_0i+p_3j-p_2k, -p_2-p_3i+p_0j+p_1k\rangle = 0$, etc., so X, Y, Z are mutually orthogonal.

Now, the corresponding flows are

$$\varphi_t(p) = p(\cos t + i\sin t), \quad \psi_t(p) = p(\cos t + j\sin t), \quad \xi_t(p) = p(\cos t + k\sin t)$$

Then

$$[X,Y] = L_X Y = \frac{d}{dt}\Big|_{t=0}\left(d\varphi_{-t}(Y_{\varphi_t(p)}) - Y_p\right)$$

Now

$$d\varphi_{-t}(Y) = d\varphi_{-t}(Y_{p(\cos t + i\sin t)}) = d\varphi_{-t}(p(\cos t + i\sin t)j) = d\varphi_{-t}(p(j\cos t + k\sin t))$$

And can check that

$$d\varphi_{-t} = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}$$

So

$$d\varphi_{-t}(Y_{\varphi_t(p)}) = \begin{pmatrix} -p_2\cos(2t) - p_3\sin(2t) \\ -p_3\cos(2t) + p_2\sin(2t) \\ p_0\cos(2t) - p_1\sin(2t) \\ p_1\cos(2t) + p_0\sin(2t) \end{pmatrix}$$

So $\frac{d}{dt}\Big|_{t=0}\left(d\varphi_{-t}(Y_{\varphi_t(p)}) - Y_p\right) = 2\begin{pmatrix} -p_3 \\ p_2 \\ -p_1 \\ p_0 \end{pmatrix} = 2pk = 2Z_p$

Therefore, in general, $[X,Y] = 2Z$ and, by symmetry, $[Y,Z] = 2X$, $[Z,X] = 2Y$

Now, suppose $\alpha, \beta, \gamma \in \Omega^1(S^3)$ are dual to X, Y, and Z, respectively. Then

$$d\alpha(Y,Z) = (\iota_Y d\alpha)(Z) = (L_Y\alpha)(Z) - d(\iota_Y\alpha)(Z)$$
$$= (L_Y\alpha)(Z) - d(\cancel{\alpha(Y)})(Z)$$

Now, it's a general fact that $(L_U\omega)(V) = L_U(\omega(V)) - \omega([U,V])$ for any $\omega \in \Omega^1(M)$ & $U, V \in \mathfrak{X}(M)$

So in our case, get $L_Y(\alpha(Z)) - \alpha([Y,Z]) = 0 - \alpha(2X) = -2$

Get 0 otherwise, so we see that $d\alpha = -2\beta\wedge\gamma$.

(Note: This means $\alpha\wedge d\alpha = 2\alpha\wedge\beta\wedge\gamma$, which is a nonzero multiple of the volume form $\alpha\wedge\beta\wedge\gamma$, so α is a contact form on S^3... in fact, it's the standard (negative) contact form on S^3)

The usual contact form on S^3 is the 1-form dual to the vector field V given by

$$V_p = ip$$

In the coords. (x_1, y_1, x_2, y_2) on $\mathbb{R}^4 = \mathbb{C}^2$,

$$V = x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2},$$

so the dual form is

$$x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

which is the usual way one sees it in contact geometry surveys.

In these coords., the form α from above is

$$x_1 dy_1 - y_1 dx_1 - x_2 dy_2 + y_2 dx_2$$