Math 8250: Day 11

Note: Corrected typos in #2 on HW.

Df: The exterior derivative $d: \Omega^*(M) \to \Omega^*(M)$ is the unique anti-derivation of degree $+1$ s.t.

① df is the differential of f $\forall f \in C^\infty(M) = \Omega^0(M)$

② $d \circ d = 0$.

Proof: In local coords, $\omega_p = \sum f_I dx_I$, so we will define:

$$(d\omega)_p = \sum \big((df_I) \wedge dx_I + f_I\, d(dx_I)\big) = \sum (df_I) \wedge d\omega_I$$

b/c $d(dx_i) = (d \circ d) x_i = 0$ for all i, so $d(dx_I) = 0$ for all I by iterating using the anti-derivation property

$$d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \cancel{d(dx_{i_1})} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} - dx_{i_1} \wedge \cancel{d(dx_{i_2})} \wedge \cdots \wedge dx_{i_k} + \cdots + (-1)^{k-1} dx_{i_1} \wedge \cdots \wedge \cancel{d(dx_{i_k})}$$

$$= 0.$$

Then independence of coords will follow from uniqueness, so just need to prove

(i) it's an antiderivation

(ii) it satisfies the cocycle condition ②

(iii) uniqueness.

(i) & (ii) are both computations. Here's the computation for the cocycle condition:

If $\omega_p = \sum f_I dx_I$, then

$$d\omega_p = \sum df_I \wedge dx_I = \sum_I \Big(\sum_i \frac{\partial f_I}{\partial x_i} dx_i\Big) \wedge dx_I$$

Then

$$d(d\omega)_p = \sum_I \sum_i \Big(\sum_j \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j \wedge dx_i\Big) \wedge dx_I = 0$$

b/c $\frac{\partial^2 f_I}{\partial x_j \partial x_i} = \frac{\partial^2 f_I}{\partial x_i \partial x_j}$, but $dx_j \wedge dx_i = -dx_i \wedge dx_j$.

For uniqueness, suppose $\exists$ degree 1 anti-derivation $D: \Omega^*(M) \to \Omega^*(M)$ satisfying ① & ②. In particular,

$Df = df$ $\forall f \in C^\infty(M)$.

Then

$$D(dx_I) = D(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_j (-1)^{j-1} dx_{i_1} \wedge \cdots \wedge D dx_{i_j} \wedge \cdots \wedge dx_{i_k} = 0$$

b/c $D(dx_i) \overset{①}{=} D(Dx_i) \overset{②}{=} 0$ by hypothesis.

Therefore, if $\omega_p = \sum f_I dx_I$,

$$(D\omega)_p \overset{\text{antiderivation prop.}}{=} \sum_I \left(Df_I \wedge dx_I + f_I \cancel{D(dx_I)}\right)$$

$$\overset{②}{=} \sum df_I \wedge dx_I$$

$$= (d\omega)_p$$

True for all $p \in M$, so $D\omega = d\omega$. ∎

Ex: If $\omega = f\,dx + g\,dy + h\,dz \in \Omega^1(\mathbb{R}^3)$, then

$$d\omega = \left(\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)\wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial z}dz\right)\wedge dy + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)\wedge dz$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)dx\wedge dz + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)dy\wedge dz$$

OTOH, $\omega = f\,dx + g\,dy + h\,dz \longleftrightarrow v = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$

and $\nabla \times v = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\frac{\partial}{\partial z}$

$$\updownarrow$$

$$d\omega = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)dy\wedge dz + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)dx\wedge dz + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy$$

So d really does capture the curl.

Likewise, if $\eta = a\,dy\wedge dz - b\,dx\wedge dz + c\,dx\wedge dy \longleftrightarrow u = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$

$$d\eta = \frac{\partial a}{\partial x}dx\wedge dy\wedge dz - \frac{\partial b}{\partial y}dy\wedge dx\wedge dz + \frac{\partial c}{\partial z}dz\wedge dx\wedge dy$$

$$= \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)dx\wedge dy\wedge dz \longleftrightarrow \nabla\cdot u = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$$

We've now succeeded in defining everything in the following diagram & showed that it commutes:

$$
\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M) \\
C^\infty(M) \arrow[r, "\nabla"] \arrow[u, equal] & \mathfrak{X}(M) \arrow[r, "\nabla\times"] \arrow[u, "\flat", "\wr"'] & \mathfrak{X}(M) \arrow[r, "\nabla\cdot"] \arrow[u, "*\flat", "\wr"'] & C^\infty(M) \arrow[u, "\wr"']
\end{tikzcd}
$$