Math 8250: Day 10

Remember that a k-form is a smooth section of $\Lambda^k(M) = \bigcup_{p\in M} \Lambda^k((T_pM)^*)$.

We denote the space of k-forms on M by $\Omega^k(M)$ and let $\Omega^*(M) = \bigoplus_k \Omega^k(M)$.

Let (U,φ) be a coord. chart in a nbhd of $p\in M^n$. For each $i=1,\dots,n$, define the map $x_i : \varphi(U)\to U$ by

$$x_i(\varphi(a_1,\dots,a_n)) = a_i$$

i.e., projection to the i^{th} local coord. Then, $\{\frac{\partial}{\partial x_i}\}$ is the basis for T_pM associated to φ and

$\{dx_i\}$ gives a basis for the dual space $(T_pM)^*$, where, by defn of the differential, $dx_i\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}(x_i) = \delta_{ij}$.

Then, using properties of the exterior product, $\{dx_{i_1}\wedge\cdots\wedge dx_{i_k} \mid i_1 < \dots < i_k\}$ is a basis for $\Lambda^k((T_pM)^*)$ and so,

for $\omega\in\Omega^k(M)$, $\omega_p = \sum a_I dx_I$ where $I = (i_1,\dots,i_k)$ & $dx_I = dx_{i_1}\wedge\cdots\wedge dx_{i_k}$.

Ex: On $\mathbb{R}^2$, have global coords., so any 1-form looks like $\omega = f\,dx + g\,dy$. So, if

$\omega = f\,dx + g\,dy$
$\eta = h\,dx + k\,dy$

Then $\omega\wedge\eta = (f\,dx+g\,dy)\wedge(h\,dx+k\,dy) = (fk-gh)\,dx\wedge dy$

So $\omega\wedge\eta(V,W) = \omega\wedge\eta\left(v_1\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}, w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}\right)$

$$= (fk-gh)(v_1w_2 - v_2w_1).$$

On $\mathbb{R}^3$, $\Omega^1(\mathbb{R}^3) = \{a\,dx + b\,dy + c\,dz\}$, $\Omega^2(\mathbb{R}^3) = \{e\,dx\wedge dy + f\,dx\wedge dz + g\,dy\wedge dz\}$

so, for $\omega\in\Omega^1(\mathbb{R}^3)$, $\eta\in\Omega^2(\mathbb{R}^3)$, $\omega\wedge\eta = (a\,dx+b\,dy+c\,dz)\wedge(e\,dx\wedge dy + f\,dx\wedge dz + g\,dy\wedge dz)$

$$= (ag - bf + ce)\,dx\wedge dy\wedge dz$$

and, for $\omega_1,\omega_2\in\Omega^1(M)$ w/ $\omega_i = a_i\,dx + b_i\,dy + c_i\,dz$,

$$\omega_1\wedge\omega_2 = (a_1dx+b_1dy+c_1dz)\wedge(a_2dx+b_2dy+c_2dz) = a_1b_2\,dx\wedge dy + a_1c_2\,dx\wedge dz + b_1a_2\,dy\wedge dx + b_1c_2\,dy\wedge dz$$
$$+ c_1a_2\,dz\wedge dx + c_1b_2\,dz\wedge dy$$
$$= (a_1b_2 - a_2b_1)\,dx\wedge dy + (a_1c_2 - a_2c_1)\,dx\wedge dz + (b_1c_2 - b_2c_1)\,dy\wedge dz$$

Now, to talk about the exterior derivative, we need to get into some details about derivations

Def: An endomorphism A of $\Lambda(V)$ is

- a derivation if $A(u\wedge v) = Au\wedge v + u\wedge Av$
- an anti-derivation if $A(u\wedge v) = Au\wedge v + (-1)^p u\wedge Av$ for $u\in\Lambda^p(V)$ $\left(\Rightarrow A(v_1\wedge\ldots\wedge v_k) = \sum_i (-1)^{i+1} v_1\wedge\ldots\wedge Av_i\wedge\ldots\wedge v_k\right)$
- of degree k if $A:\Lambda^i(V)\to\Lambda^{i+k}(V)$ for all i.

Ex: For $u\in\Lambda(V)$, define the interior product with u, denoted ι_u, as an endomorphism of $\Lambda(V)^*$

by: $$(\iota_u v^*, w) = (v^*, u\wedge w)$$

Notice that, if $m_u:\Lambda(V)\to\Lambda(V)$ is the endomorphism given by $m_u(v) = u\wedge v$, then $m_u^* = \iota_u$.

Prop: If $u\in V = \Lambda^1(V)$, then ι_u is an anti-derivation of degree -1.

Proof: Obviously degree -1 since m_u is degree 1 on $\Lambda(V)$.

Now, let $v^* = v_1^*\wedge\ldots\wedge v_k^*\in\Lambda^k(V)^*$, $w = w_2\wedge\ldots\wedge w_k\in\Lambda^{k-1}(V)$.

Then
$$(\iota_u v^*, w) = (v^*, u\wedge w) = (v^*, u\wedge w_2\wedge\ldots\wedge w_k) = \det\left(v_i^*(w_j)\right)$$

if I let $w_1 = u$.

OTOH,
$$\left(\sum (-1)^{i+1} v_1^*\wedge\ldots\wedge \iota_u v_i^*\wedge\ldots\wedge v_k^*, w_2\wedge\ldots\wedge w_k\right) = \sum (-1)^{i+1} v_i^*(u)\left(v_1^*\wedge\ldots\wedge\widehat{v_i^*}\wedge\ldots\wedge v_k^*, w_2\wedge\ldots\wedge w_k\right)$$
$$= \sum (-1)^{i+1} v_i^*(u)\det\left(v_l^*(w_j)\right)_{\substack{l\neq i\\ j\neq 1}} = \det\left(v_k^*(w_j)\right)$$

Df: The exterior derivative $d:\Omega^*(M)\to\Omega^*(M)$ is the unique anti-derivation of degree $+1$ s.t.

① df is the differential of f $\forall f\in C^\infty(M)=\Omega^0(M)$

② $d\circ d=0$.

Idea: In local coords, $\omega_p=\sum f_I dx_I$, so

$$(d\omega)_p=\sum\big((df_I)\wedge d\omega_I+f_I\, d(d\omega_I)\big)=\sum(df_I)\wedge d\omega_I$$

b/c $d(dx_i)=(d\circ d)x_i=0$ for all i, so $d(d\omega_I)=0$ for all I by iterating using the anti-derivation property

$$d(dx_{i_1}\wedge\cdots\wedge dx_{i_k})=\cancel{d(dx_{i_1})}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_k}-dx_{i_1}\wedge \cancel{d(dx_{i_2})}\wedge\cdots\wedge dx_{i_k}+\cdots+(-1)^{k-1}dx_{i_1}\wedge\cdots\wedge \cancel{d(dx_{i_k})}$$

$$=0.$$

Then just need to show independence of coords. & uniqueness.