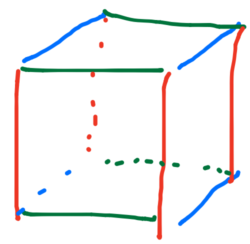
Math 8250: Day 1

Consider $M = S^3$ or $M = T^3 = S^1 \times S^1 \times S^1$.

Suppose V is a divergence-free vector field on M.



Q: Does there exist a vector field U on M so that $V = \nabla \times U$?

A: If $M = S^3$, then yes. If $M = T^3$, then it depends. For example, if $V = (0,0,1)$, then certainly $\nabla \cdot V = 0$, but V is not the curl of any field.

Why?

This follows from a special case of the Hodge Decomposition Theorem. To set notation, let $VF(M)$ be the space of C^∞ vector fields on the compact, oriented 3-manifold M without boundary. Then

Hodge thm for 3-mflds: The vector space $VF(M)$ has the direct sum decomposition

$$VF(M) = FK(M) \oplus HK(M) \oplus G(M)$$

where

$$FK(M) = \{V : \nabla \cdot V = 0,\ V \text{ fluxless}\} = \text{"fluxless knots"}$$

$$HK(M) = \{V : \nabla \cdot V = 0,\ \nabla \times V = 0\} = \text{"harmonic knots"}$$

$$G(M) = \{V : \exists f \in C^\infty(M) \text{ s.t. } \nabla f = V\} = \text{"gradients"}$$

where a vector field being fluxless means its flux through any closed, embedded surface in M is zero.

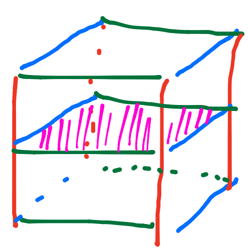
Moreover, the above direct sum is orthogonal w.r.t. the L^2 inner product on $VF(M)$, $HK(M)$ is finite-dimensional, and the following hold:

$$\ker(\nabla\times) = HK(M) \oplus G(M)$$
$$\operatorname{im}(\nabla\times) = FK(M)$$
$$\ker(\nabla\cdot) = FK(M) \oplus HK(M)$$
$$\operatorname{im}(\nabla) = G(M)$$

Finally, $HK(M)$ is isomorphic (as a vector space) to the first (singular) cohomology group of M with real coefficients:

$$HK(M) \cong H^1(M;\mathbb{R}).$$

Clearly, this answers the question: a divergence-free field V on M is a curl if & only if it is fluxless. In the case of S^3, $H^1(S^3;\mathbb{R}) = \{0\}$, so all divergence-free fields are fluxless, whereas in the case of T^3 it's clear that the field $V=(0,0,1)$ has nonzero flux through the closed surface



In fact, this field V is clearly curl-free and hence an element of $HK(M)$.

Finally, recall that the vector Laplacian Δ is given by

$$\Delta V = -\nabla\times\nabla\times V + \nabla(\nabla\cdot V)$$

Prop: ① $\Delta: FK(M) \to FK(M)$ is bijective

② $\Delta: G(M) \to G(M)$ is bijective

Cor: $\ker \Delta = HK(M)$, hence the word "harmonic" in "harmonic knots"

Though it will be beyond the scope of this class (see Schwarz's book Hodge Decomposition - A Method for Solving Boundary Value Problems for details) there is an analogous result for compact domains in 3-space that, for example, tells you when a divergence-free vector field is a curl (i.e., has a vector potential) and when a curl-free field is a gradient (i.e. has a scalar potential). As you can imagine, this is hugely important in electromagnetism & fluid dynamics.

However, this is not only (or even primarily) a class about 3-manifolds & vector fields. Instead, it is a class about n-manifolds & differential forms. I assume some familiarity w/ manifolds, but we'll do some review in the first week or two. Then we'll spend some time on forms, but let me give you a sense now.

A differential k-form ω on a manifold M is an alternating, multilinear map

$$\omega : \underbrace{VF(M) \times \ldots \times VF(M)}_{k \text{ copies}} \to C^\infty(M)$$

Let $\Omega^k(M)$ be the set of k-forms on M.

When $k=0$, a 0-form is just a smooth function on M, so $\Omega^0(M) = C^\infty(M)$.

When $k=1$, a 1-form ω is just a linear functional on vector fields: if $V \in VF(M)$, then at each point $x \in M$ the form ω assigns to V the number $\omega_x(V)$ in a smoothly-varying way. Clearly, there's a duality b/w vector fields & 1-forms: on $\mathbb{R}^n$, if $e_1, \ldots, e_n$ is the std. basis, then there's the corresponding basis $dx_1, \ldots, dx_n$ of 1-forms on $\mathbb{R}^n$ s.t. $dx_i(e_j) = \delta_{ij}$.

Then $\Omega^1(M) \cong VF(M)$

When $k=n$, the dimension of M, the only alternating, multilinear maps on a maximal set of linearly independent vectors are multiples of the determinant, so $\Omega^n(M) \cong C^\infty(M)$.

Notice that $\Omega^0(M) \cong \Omega^n(M)$. In general, it will turn out that there is a "natural" isomorphism

$$\Omega^k(M) \cong \Omega^{n-k}(M) \quad \text{for all } k=0,1,\dots,n.$$

Finally (for now), there is a differentiation operator d that sends k-forms to $(k+1)$-forms for all k and is called the exterior derivative. Moreover, d is a (co)chain map: $d \circ d = 0$. Therefore,

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

is a (co)chain complex & we can take its (co)homology. This yields the so-called de Rham cohomology groups, which (IMPORTANT THEOREM) turn out to be iso. to the usual singular cohomology groups w/ real coefficients.

We have the following diagram relating vector fields & forms on 3-mfds:

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M) \\
C^\infty(M) \arrow[u, "\cong"] \arrow[r, "\nabla"] & VF(M) \arrow[u, "\cong"] \arrow[r, "\nabla\times"] & VF(M) \arrow[u, "\cong"] \arrow[r, "\nabla\cdot"] & C^\infty(M) \arrow[u, "\cong"]
\end{tikzcd}$$

Hodge Decomposition Theorem: Let M^n be a smooth, closed, oriented, Riemannian mfd of dimension n. Then for all $0 \le k \le n$,

$$\Omega^k(M) = \delta\Omega^{k+1}(M) \oplus \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M)$$

where $\delta: \Omega^{k+1}(M) \to \Omega^k(M)$ is the L^2 adjoint of d and

$$\mathcal{H}^k(M) := \{\omega : d\omega = 0, \delta\omega = 0\}.$$

Moreover, $\mathcal{H}^k(M)$ is finite-dim'l & isomorphic to $H^k(M;\mathbb{R})$.