Math 8250 HW #5

Due 11:15 AM Friday, March 8

1. In class we (implicitly) showed that Stokes' Theorem implies the Divergence Theorem from vector calculus, which says that, if $\vec{F}$ is a smooth vector field on a compact domain $D \subset \mathbb{R}^3$ with smooth boundary $\Sigma = \partial D$, then

$$\int_D \nabla \cdot \vec{F}\, dx\, dy\, dz = \int_\Sigma \vec{F} \cdot \vec{n}\, dA,$$

   where $\vec{n} = (n_1, n_2, n_3)$ is the unit outward normal vector to the surface Σ and the form $dA = n_1\, dy \wedge dz + n_2\, dz \wedge dx + n_3\, dx \wedge dy$ is the area form on Σ (the fact that this is the area form follows from Problem 5(a) below).

   Here's a situation in which the Divergence Theorem has an interesting and perhaps surprising consequence in electrostatics. Assume the origin is contained in the interior of D and an electric charge of magnitude q is placed at the origin. The resulting force field $\vec{F}$ on D is given by $q\frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r}$ is the radial field (x, y, z). Show that the amount of charge q can be determined from the force on the boundary by proving Gauss's law, which says that

$$\int_\Sigma \vec{F} \cdot \vec{n}\, dA = 4\pi q.$$

2. Prove that $H^1_{\mathrm{dR}}(S^2)$, the 1-dimensional de Rham cohomology group of the 2-sphere, is trivial (in other words, all closed 1-forms on S^2 are exact).

3. Let M^n be closed and orientable. Show that for any $\omega \in \Omega^{n-1}(M)$, the n-form $d\omega$ is zero at some point in M (in other words, volume forms on closed, oriented manifolds *cannot* be exact, and hence the top-dimensional de Rham cohomology group of such a manifold cannot be trivial).

4. (a) Suppose that M^n is a closed manifold such that $M = \partial W$ for some oriented $(n+1)$-manifold W and that $f : M \to N$ is a smooth map. Let $\omega \in \Omega^n(N)$ be closed. Show that if the map f extends to all of W, then $\int_M f^*\omega = 0$.

   (b) Use part (a) to show that, if $f_0, f_1 : M \to N$ are homotopic, then

$$\int_M f_0^*\omega \;= \int_M f_1^*\omega$$

   for any closed $\omega \in \Omega^n(N)$.

5. Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersion and let M^n be given the Riemannian metric induced by the immersion, meaning that for any $p \in M$ and $v, w \in T_pM$,

$$\langle v, w \rangle_p := \langle df v, df w \rangle_{f(p)}.$$

   Now, suppose that M is oriented and that ν is the oriented unit normal vector field along $f(M)$, so that $\nu, df v_1, \ldots, df v_n$ gives the standard orientation on $T_{f(p)}\mathbb{R}^{n+1}$ whenever $v_1, \ldots, v_n$ gives the orientation on T_pM.

(a) Show that the Riemannian volume form on M is given by

$$\mathrm{dVol}_M = f^* \iota_\nu dx_1 \wedge \ldots \wedge dx_{n+1}.$$

(b) Find the Riemannian area form on the torus of revolution parametrized by

$$((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi).$$