Math 8250 Final Exam

Due 3:00 PM Tuesday, May 7

Ground rules: this is an exam, so the rules are a bit different than for homework. You may not collaborate with other people in the class or with anybody else: you should not be discussing specifics of the problems with anybody other than me. Also, you may not use any online resources (this means no Googling!). You are more than welcome to use the lecture notes from class and Warner's book. If you have another book that you like, you're also welcome to use that as long as you cite it, but you should not be Googling for books that might be helpful.

1. Consider the following subset of $SO(3)$:

   $$S := \{A \in SO(3) : A = A^\top,\ A \neq I\},$$

   where I is the 3×3 identity matrix.

   Show that S is a smooth submanifold of $SO(3)$. What familiar manifold is it? What happens if I add the identity matrix to S?

2. Show that for every nonzero $\omega \in \Omega^2(\mathbb{R}^6)$ there is some basis $\omega_1, \ldots, \omega_6$ for $\Omega^1(\mathbb{R}^6)$ (thought of as a $C^\infty(\mathbb{R}^6)$-module) so that ω can be written in one of the following forms:

   $$\omega_1 \wedge \omega_2, \quad \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4, \quad \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 + \omega_5 \wedge \omega_6.$$

   Show that these three possibilities are mutually incompatible (e.g., if $\omega = \omega_1 \wedge \omega_2$ in some basis, there is no basis $\eta_1, \ldots, \eta_6$ so that $\omega = \eta_1 \wedge \eta_2 + \eta_3 \wedge \eta_4$). (Hint: think about $\omega \wedge \omega$.)

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a rotation (i.e., $f \in SO(n)$). Then f induces a map on S^{n-1}, the unit sphere in $\mathbb{R}^n$. Show that f is a volume-preserving diffeomorphism of S^{n-1}, meaning that f is a diffeomorphism and that $f^* \,\mathrm{dVol}_{S^{n-1}} = \mathrm{dVol}_{S^{n-1}}$ where $\mathrm{dVol}_{S^{n-1}}$ is the standard volume form on S^{n-1}. (Hint: Exercise #5 from HW 5 may come in handy.)

4. Suppose M^n and N^n are closed manifolds of the same dimension and that $f : M \to N$ is a smooth map. Pick some $\omega_0 \in \Omega^n(N)$ so that $\int_N \omega_0 \neq 0$. We define the *degree* of the map f, denoted $\deg(f)$, by the equation

   $$\int_M f^*\omega_0 = \deg(f) \int_N \omega_0.$$

   (a) Show that the degree of f is well-defined (i.e., independent of the choice of ω_0).

   (b) Show that if f and g are homotopic maps from M to N, then $\deg(f) = \deg(g)$.

   (c) Show that there does not exist a nowhere-vanishing vector field on S^n when n is even. (Hint: first, use part (b) to show that the antipodal map is not homotopic to the identity map. Then show that a nowhere-vanishing vector field implies the existence of a homotopy from the antipodal map to the identity map.)

5. Suppose $F : M^m \to N^n$ is smooth and has no critical points. Define the distribution $\mathfrak{D}$ on M by $\mathfrak{D}(x) = \{V \in T_pM : df_x(V) = 0\}$ for each $x \in M$. Show that $\mathfrak{D}$ is involutive. What are the maximal integral manifolds of $\mathfrak{D}$?

6. Let M^n be a closed Riemannian manifold and let Δ be the Laplacian on p-forms. Then a non-zero $\omega \in \Omega^p(M)$ such that $\Delta\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}$ is called an *eigenform* of Δ corresponding to the eigenvalue λ. As you would expect, the space of forms corresponding to a given eigenvalue is called the *eigenspace* of that eigenvalue. For example, the eigenspace of the eigenvalue 0 is precisely the space of harmonic forms on M.

   (a) Show that all the eigenvalues of Δ are non-negative.

   (b) Show that each of the eigenspaces of Δ are finite-dimensional.

   (c) Show that the eigenspaces corresponding to different eigenvalues are perpendicular (with respect to the L^2 inner product on $\Omega^p(M)$).

   (d) Show that the eigenvalues of Δ have no finite accumulation point.

   These facts, along with the more challenging result that there actually *are* non-zero eigenvalues of Δ, serve as the basis for Fourier analysis of forms on manifolds.