Math 676 Discussion Problems #4

1. Suppose that the Lie group G acts in a Hamiltonian way on the symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with moment maps $\mu_1 : M_1 \to \mathfrak{g}^*$ and $\mu_2 : M_2 \to \mathfrak{g}^*$. Show that the diagonal G action on $M_1 \times M_2$ (with the obvious symplectic form $\pi_1^*\omega_1 + \pi_2^*\omega_2$) is Hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2).$$

2. (Feel free to skip this and come back to it after completing the other problems, since it is the most time-consuming) The unitary group $U(n)$ acts on $(\mathbb{C}^n, \omega_0)$, where $\omega_0 = \frac{i}{2}\sum dz \wedge d\bar{z}$, in the obvious way. Show that this action is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ given by

$$\mu(\vec{z}) = \frac{i}{2}\vec{z}\vec{z}^*,$$

where $\vec{z}^*$ is the conjugate transpose of $\vec{z}$, and we've identified $\mathfrak{u}(n)^*$ with the space of skew-Hermitian matrices $\mathfrak{u}(n)$ via the inner product $\langle A, B\rangle = \mathrm{trace}(A^*B)$.

3. Consider the usual action of $U(k)$ on $\mathbb{C}^{k\times n}$, the space of complex $k \times n$ matrices. Identifying $\mathfrak{u}(k)^*$ with $\mathfrak{u}(k)$ as in the previous problem, show that the $U(k)$ action is Hamiltonian with moment map $\mu : C^{k\times n} \to \mathfrak{u}(k)$ given by

$$\mu(A) = \frac{i}{2}AA^* + \frac{1}{2i}I_k,$$

where I_k is the $k \times k$ identity matrix.

4. With the action and moment map as in the previous problem, determine $\mu^{-1}(0)$ and use this to show that the symplectic reduction $\mathbb{C}^{k\times n} /\!\!/_0\, U(k)$ is precisely $G_k(\mathbb{C}^n)$, the Grassmannian of complex k-dimensional linear subspaces of $\mathbb{C}^n$.

   In particular, this proves the fact stated in class that the symplectic reduction (or GIT quotient) of $\mathbb{C}^{2\times n}$ by the obvious $U(2)$ (or $GL(2, \mathbb{C})$) action is the Grassmannian $G_2(\mathbb{C}^n)$. (Taking the reduction of this Grassmannian by $U(1)^n/U(1)$ then gets back to the polygon space $\mathrm{Pol}(n)$.)

5. Recall that $\mathbb{CP}^1$ is diffeomorphic to S^2; however, from the perspective of complex geometry the natural symplectic form on $\mathbb{CP}^1$ is not the standard symplectic form $\omega_{\text{std}} = d\theta \wedge dz$ on S^2, but rather the *Fubini–Study form* ω_{FS}, which in the local coordinate $\frac{z_1}{z_0} = z = x + iy$ on the coordinate patch $\{[z_0 : z_1] \in \mathbb{CP}^1 : z_0 \neq 0\}$ has the form

$$\omega_{\text{FS}} = \frac{i}{2}\partial\bar{\partial}\log(z\bar{z} + 1) = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2} = \frac{1}{4}\omega_{\text{std}}.$$

   Determine the moment polytope of the T^2 action on $(\mathbb{CP}^1 \times \mathbb{CP}^1, \pi_1^*\omega_{\text{FS}} + \pi_2^*\omega_{\text{FS}})$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot ([z_0 : z_1], [w_0 : w_1]) = ([z_0 : e^{i\theta_1}z_1], [w_0 : e^{i\theta_2}w_1]).$$