Math 676 Discussion Problems #3

1. Consider a collection of n points $p_1, p_2, \ldots, p_n$ on $\mathbb{P}^1$. The goal is to understand when this collection is (semi-)stable with respect to the $PSL(2,\mathbb{C})$ action on $\mathbb{P}^1$. In order to get a concrete linearization of this action, we identify n points on $\mathbb{P}^1$ with the unique (up to scale) homogeneous polynomial of degree n with the points as its roots. (Strictly speaking this means we're considering points in the symmetric product $\Sigma^n\mathbb{P}^1$ rather than $(\mathbb{P}^1)^n$.) The collection of degree-n homogeneous polynomials of degree n up to scale (i.e., the projectivization of $H^0(\mathcal{O}_{\mathbb{P}^1}(n))$) has a natural line bundle, where the fiber over each point is all possible scalings of a representative.

   (a) Make sense of the above, if it's not clear.

   Now, the goal is to show that $(p_1, \ldots, p_n)$ is semi-stable if and only if no more than $n/2$ of the p_i's are equal, and stable if and only if fewer than $n/2$ of the p_i's are equal.

   Here's a sketch of a strategy:

   (b) Use the Hilbert–Mumford criterion, so choose a particular 1-parameter subgroup $\mathbb{C}^* \leq PSL(2,\mathbb{C})$.

   (c) Let f be a degree-n homogeneous polynomial, and determine $f_0 := \lim_{\lambda\to 0} \lambda \cdot f$. (*Hint:* try diagonalizing the 1-parameter subgroup and writing f as a sum of monomials)

   (d) Determine the weight $\rho(f)$ of the action on the fiber (i.e., line) over f_0 and determine the conditions when it is negative, zero, or positive.

   (e) This was for one particular 1-parameter subgroup, what about all possible 1-parameter subgroups?

2. Now we use the Deligne–Mostow weighted quotient. Recall that the *nice semi-stable points* on $(S^2)^n$ are those which are either stable or whose $PSL(2,\mathbb{C})$-orbit is closed in the set $M_{sst} \subset (S^2)^n$ of semi-stable points. Also, recall that the strictly semi-stable points $M_{cusp} = M_{sst} \backslash M_{st}$ correspond to partitions $S_1 \sqcup S_2$ of $\{1, \ldots, n\}$ such that $S_1 = \{i_1, \ldots, i_k\}$, $S_2 = \{j_1, \ldots, j_{n-k}\}$ and $r_{i_1} + \ldots + r_{i_k} = 1$, and either

$$e_{i_1} = e_{i_2} = \ldots = e_{i_k} \quad \text{or} \quad e_{j_1} = e_{j_2} = \ldots = e_{j_{n-k}}.$$

   Show that a point is in $M_{cusp} \cap M_{nsst}$ if and only if both of the above equalities hold.

3. Recall that $Q_{cusp} = M_{cusp}/\sim$, where $\sim$ is the equivalence relation defined in class. Show that each equivalence class in Q_{cusp} contains a unique $PSL(2,\mathbb{C})$-orbit of nice semi-stable points.