Math 676 Discussion Problems #2

1. Consider $T^2 = S^1 \times S^1$ with the symplectic form $\omega = d\theta_1 \wedge d\theta_2$. I claimed in class that the symplectic vector fields $V_i := \frac{\partial}{\partial \theta_i}$ for $i = 1, 2$ are *not* Hamiltonian; in other words, that

$$\iota_{V_1}\omega \quad \text{and} \quad \iota_{V_2}\omega$$

are not exact, and hence represent nontrivial classes in the de Rham cohomology group $H^1_{\mathrm{dR}}(T^2)$. Prove this. (*Hint:* Use Stokes' theorem)

2. Recall that the standard symplectic form on $\mathbb{C}^n$ is $\frac{i}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$. Consider the function $H : \mathbb{C}^n \to \mathbb{R}$ given by

$$H(z_1, \ldots, z_n) = \frac{1}{2}\sum_{i=1}^n |z_i| = \frac{1}{2}\sum_{i=1}^n z_i \bar{z}_i.$$

What is the associated Hamiltonian vector field X_H? What are the trajectories of X_H?

3. Let M be a smooth manifold (not necessarily symplectic) and let α and ω be the tautological 1-form and the canonical symplectic form on the cotangent bundle T^*M, respectively. Recall that a 1-form $\eta \in \Omega^1(M)$ is defined to be a smooth section of T^*M. Define

$$M_\eta := \eta(M) = \{(p, \eta_p) : p \in M\} \subset T^*M$$

to be the image of the section. Prove that M_η is Lagrangian (meaning that $\omega|_{M_\eta} \equiv 0$) if and only if η is closed.

4. Suppose M is an n-dimensional manifold, $V \in \mathfrak{X}(M)$ is a vector field, and $\alpha \in \Omega^k(M)$ is a k-form. The goal of this exercise is to prove Cartan's magic formula

$$\mathcal{L}_V \alpha = \iota_V d\alpha + d\iota_V \alpha. \qquad (\star)$$

   (a) Prove that ($\star$) holds for 0-forms (i.e., smooth functions).

   (b) Show that both sides of ($\star$) commute with d.

   (c) Prove that both sides of ($\star$) are derivations of the algebra $(\Omega^*(M), \wedge)$ of forms. For example, you will need to show that

$$\mathcal{L}_V(\alpha \wedge \beta) = (\mathcal{L}_V \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_V \beta).$$

   (d) Suppose (φ, U) is a local coordinate chart on M, so that $U \subset \mathbb{R}^n$ is open and $\varphi : U \to M$ is a homeomorphism onto its image $\varphi(U)$. Show that $\Omega^*(\varphi(U), \wedge)$ is generated as an algebra by $\Omega^0(\varphi(U))$ and $d\Omega^0(\varphi(U))$ (i.e., the exact 1-forms). In other words, form on $\varphi(U)$ is a linear combination of wedge products of elements of $\Omega^0(\varphi(U))$ and elements of $d\Omega^0(\varphi(U))$.

   (e) Use the preceding parts to conclude ($\star$).