Math 676 Discussion Problems #1

Proposition 0.1. *Let ω be a skew-symmetric bilinear map on a vector space V. Then there is a basis $u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n$ for V such that*

$$\begin{cases} \omega(u_i, v) = 0 & \textit{for all } i \textit{ and all } v \in V \\ \omega(e_i, e_j) = 0 = \omega(f_i, f_j) & \textit{for all } i, j \\ \omega(e_i, f_j) = \delta_{ij} & \textit{for all } i, j \end{cases}$$

This is proved by a sort of skew-symmetric version of Gram–Schmidt; see the first section of Ana Cannas da Silva's book.

1. Let V be a vector space. Recall that $\bigwedge^k(V^*)$ is the kth exterior power of the dual space of V, elements of which can be identified with alternating multilinear maps on V.
   (a) Show that any $\omega \in \bigwedge^2(V^*)$ can be written as $\omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*$ with respect to the dual basis to the basis from the above proposition. Note that $\dim V = 2n + k$.
   (b) In this language, a symplectic map $\omega : V \times V \to \mathbb{R}$ is just a nondegenerate element of $\bigwedge^2(V^*)$.
   Show that if (V, ω) is symplectic with V being $2n$-dimensional, then the nth exterior power $\omega^n = \omega \wedge \ldots \wedge \omega \in \bigwedge^n(V^*)$ does not vanish.
   (c) Conversely, given $\omega \in \bigwedge^2(V^*)$ such that ω^n is nonvanishing, show that ω is symplectic.
   (d) Conclude that if (M^{2n}, ω) is a symplectic manifold if and only if $\omega \in \Omega^2(M)$ is closed and $\omega^n \in \Omega^{2n}(M)$ is a volume form.

2. Let (M^{2n}, ω) be symplectic and let ω^n be the volume form induced by the symplectic form.
   (a) Show that if M is compact, then ω^n is not exact (i.e., it is not the exterior derivative of some $(2n-1)$-form on M). In other words, $[\omega^n]$ is a nontrivial de Rham cohomology class. (*Hint: Stokes' Theorem*)
   (b) Conclude that $[\omega]$ is also a nontrivial de Rham cohomology class; in particular, ω is not exact.

3. Recall the standard area form ω on the unit 2-sphere given by

$$\omega_p(u, v) = (u \times v) \cdot p.$$

   Show that in cylindrical coordinates (θ, z) on S^2 away from the north and south poles (we omit r, since on the unit sphere $r^2 = 1 - z^2$), where $0 \leq \theta < 2\pi$ and $-1 \leq z \leq 1$, we can write the standard area form ω as

$$\omega = d\theta \wedge dz.$$