Math 670: Day 9

Differential forms are almost tensor fields, but we need to take an appropriate quotient to incorporate the alternating condition.

Specifically, just as the tensor product is the way to turn multilinear maps into linear maps, the exterior product is the way to turn alternating, multilinear maps into linear maps.

Def: Let $C(V) = \bigoplus_k V_k^0$ be the algebra of $(\cdot, 0)$ tensors on a v.s. V & let $I(V)$ be the two-sided ideal generated by elements of the form $v \otimes v$ for $v \in V$. Then the exterior algebra of V is the quotient

$$\Lambda(V) := C(V) / I(V).$$

This is a graded algebra w/ product denoted by $\wedge$. Of course, the product is just that induced by $\otimes$: the residue class of $v_1 \otimes \dots \otimes v_k$ is denoted $v_1 \wedge v_2 \wedge \dots \wedge v_k$.

The grading is given by the degree of the wedge product, since

$$\Lambda(V) = \bigoplus_{k \geq 0} \Lambda^k(V)$$

where $\Lambda^k(V) = V_k^0 / I_k$ & $I_k = I(V) \cap V_k^0$.

Prop: If $a \in \Lambda^k(V)$, $b \in \Lambda^\ell(V)$, then $b \wedge a = (-1)^{k\ell} a \wedge b$.

Proof: Exercise. ☐

Cor: $v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)} = (-1)^{\operatorname{sgn}(\sigma)} v_1 \wedge \dots \wedge v_k$ for $\sigma \in S_k$.

Prop: If $\{e_1, \dots, e_n\}$ is a basis for V, then $\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$ is a basis for $\Lambda^k(V)$. In particular,

$$\dim \Lambda^k(V) = \binom{n}{k} \text{ for } 0 \leq k \leq n \text{ \& zero otherwise, so } \dim \Lambda(V) = 2^n.$$

Proof: Exercise. ☐

This is kind of awesome b/c finite-dim'l vector spaces are always nicer than infinite-dim'l (note the tensor algebra is ∞-dim'l).

As you would expect, $\Lambda(V)$ also satisfies a universal property:

Thm (Universal Property): If $\varphi: \underbrace{V\times\cdots\times V}_{k} \to \Lambda^k(V)$ is given by $(v_1,\dots,v_k)\mapsto v_1\wedge\cdots\wedge v_k$ & if $F: V\times\cdots\times V\to U$ is an

alternating multilinear map, then $\exists!$ linear map $\widetilde{F}: \Lambda^k(V)\to U$ s.t. the diagram commutes:

$$\begin{tikzcd} V\times\cdots\times V \arrow[r, "F"] \arrow[d, "\varphi"'] & U \\ \Lambda^k(V) \arrow[ur, dotted, "\exists!\widetilde{F}"'] & \end{tikzcd}$$

Again, this property uniquely defines $\Lambda^k(V)$ up to isomorphism.

In particular, when $U=\mathbb{R}$ (or whatever the base field is), we have that $\Lambda^k(V)^*\cong A_k(V)$, the space of alternating, multilinear maps on V^k.

Now, we can define the exterior k bundle on M:

$$\Lambda^k(M) := \bigcup_{p\in M}\Lambda^k((T_pM)^*)$$

And the exterior algebra bundle over M:

$$\Lambda(M) := \bigcup_{p\in M}\Lambda((T_pM)^*)$$

Then a differential k-form on M is a smooth section of $\Lambda^k(M)$; i.e., a smooth map $\omega: M\to\Lambda^k(M)$ s.t. $\omega(p)\in\Lambda^k((T_pM)^*)$.

And now it should start to become clear that this is just another way of saying a k-form is an alternating, multilinear map $\mathfrak{X}(M)^k\to C^\infty(M)$.
(provided we can show $\Lambda^k(V)^*\cong\Lambda^k(V^*)$)

To spell it out, recall that a nondegenerate pairing of vector spaces V & W is a bilinear map

$$(\cdot,\cdot): V\times W\to\mathbb{R}$$

s.t. $W\ni w\neq 0 \Rightarrow (v,w)\neq 0$ for some $v\in V$ (& the other way).

Equivalently, a pairing is just a linear map $V\otimes W\to\mathbb{R}$, which is to say, an elt of $(V\otimes W)^*$.

Clearly, if V & W are nondegenerately paired, then $\varphi: V\to W^*$ & $\psi: W\to V^*$ are injective.
$\varphi(v)(w)=(v,w)$ $\psi(w)(v)=(w,v)$

Ex: An inner product on V is a nondegenerate pairing & this induces an injective map $V\to V^*$ (which is then an iso. if V is finite-dim'l).

Now, we can define a nondegenerate pairing $\Lambda^k(V^*)\times\Lambda^k(V)\to\mathbb{R}$ by
$(a^*,b)=\det((a_i^*(b_j))_{i,j})$ for $a^*=a_1^*\wedge\cdots\wedge a_k^*$ & $b=b_1\wedge\cdots\wedge b_k$ & extending linearly.

This then induces the isomorphism $\Lambda^k(V^*) \cong \Lambda^k(V)^* \cong A_k(V)$, & hence $\Lambda(V^*) = \bigoplus_k \Lambda^k(V^*) \cong \bigoplus_k \Lambda^k(V)^* \cong \left(\bigoplus_k \Lambda^k(V)\right)^*$

$$\cong \Lambda(V)^*$$

And now, when $V = T_pM$, we see that $\Lambda^k((T_pM)^*) \cong A_k(T_pM)$, so, extending to the bundle this proves the claim.