Math 670: Day 8

Ex: Let M^3 be a smooth 3-mfd w/ a Riemannian metric g. As before, each $X \in \mathfrak{X}(M)$ can be identified w/ a 1-form

$$\alpha =: X^\flat$$

via the musical isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ (w/ inverse $\sharp : \Omega^1(M) \to \mathfrak{X}(M)$)

The cool thing about Riemannian 3-mfds is that we have a cross product & hence a vector triple product

$$[X, Y, Z]_p = g_p(X, Y \times Z) = \det[X_p \ \ Y_p \ \ Z_p]$$

As a result, we also have an isomorphism $\star^\flat : \mathfrak{X}(M) \to \Omega^2(M)$ given by

$$\star(X^\flat)_p(Y, Z) = [X, Y, Z]_p$$

when X, Y, Z are written in orthonormal (w.r.t. g) local coords at p.

This is clearly smooth & multilinear, as well as alternating since $\star(X^\flat)_p(Y,Y) = [X,Y,Y]_p = \det[X_p \ \ Y_p \ \ Y_p] = 0$

We already saw that $\Omega^3(M) \cong C^\infty(M)$, so we get the following commutative diagram:

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M) \\
C^\infty(M) \arrow[u, equal] \arrow[r, "\nabla"] & \mathfrak{X}(M) \arrow[u, "\flat", "\cong"'] \arrow[r, "\nabla\times"] & \mathfrak{X}(M) \arrow[u, "\star^\flat", "\cong"'] \arrow[r, "\nabla\cdot"] & C^\infty(M) \arrow[u, "\cong"']
\end{tikzcd}$$

I haven't really said what d is, but the point is there's some invariant, natural way of defining the exterior derivative of a k-form for any k which exactly specializes to give the divergence & gradient on any mfd &, additionally, the curl on 3-mfds.

Hopefully the fact that div, grad, & curl are all essentially the same thing in this framework is a good advertisement for differential forms.

In fact, it gets even better: the Fundamental Theorem of Calculus, Green's Theorem, & the divergence theorem are all special cases of the same theorem: Stokes' Theorem for differential forms.

2 comments: ① In the diagram, $\nabla \times \nabla f = 0 \ \forall f \in C^\infty(M)$ & $\nabla \cdot (\nabla \times X) = 0 \ \forall X \in \mathfrak{X}(M)$, so $d \circ d = 0$ no matter whether you start in $\Omega^0(M)$ or $\Omega^1(M)$ (or, for that matter, $\Omega^2(M)$ or $\Omega^3(M)$, since the higher d's are all trivial)

In fact, in general $\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M)$ is always a chain complex.

Then, as usual for chain complexes, we take the homology, getting the k^{th} de Rham cohomology group

$$H^k_{dR}(M) := \ker d_k / \operatorname{im} d_{k+1}$$

② $\Omega^1(M) \cong \mathfrak{X}(M) \cong \Omega^2(M)$ & $\Omega^0(M) = C^\infty(M) \cong \Omega^3(M)$.

In genl, if M is an n-manifold, then $\Omega^k(M) \cong \Omega^{n-k}(M)$ $\forall$ k.

This will then specialize to a version of Poincaré duality for the de Rham cohomology gps.

Tensor Algebras & Exterior Algebras

To define differential forms properly (& to turn forms into a graded algebra), we need to do some (linear) algebra.

Given vector spaces V & W, define the tensor product $V \otimes W$ to be the vector space of elements $v \otimes w$ for $v \in V, w \in W$, s.t.

① $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

② $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

③ $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$

Ex: If $V = \mathbb{R}^m$, $W = \mathbb{R}^n$, then $\vec{v} \otimes \vec{w}$ corresponds to the $m \times n$ matrix $\vec{v}\,\vec{w}^T = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} [w_1 \cdots w_n]$

Thm (Universal Property): If $\varphi : V \times W \to V \otimes W$ is the map given by $(v,w) \mapsto v \otimes w$, & $F : V \times W \to U$ is bilinear, then $\exists!$ linear map $\tilde{F} : V \otimes W \to U$ making the diagram commute:

$$\begin{tikzcd} V \times W \arrow[r, "F"] \arrow[d, "\varphi"'] & U \\ V \otimes W \arrow[ur, dashed, "\exists!\,\tilde{F}"'] & \end{tikzcd}$$

Moreover, this uniquely characterizes $V \otimes W$: any v.s. satisfying this property is isomorphic to $V \otimes W$.

The point is that tensor products are a way of turning bilinear (or, more generally, multilinear) maps into linear maps.

Prop: If $e_1, \dots, e_m$ is a basis for V & $f_1, \dots, f_n$ is a basis for W, then $\{e_i \otimes f_j\}_{i,j}$ is a basis for $V \otimes W$.

In particular, $\dim V \otimes W = mn$.

Def: The tensor space of type (r,s) associated w/ V is the vector space

$$V_r^s = \underbrace{V\otimes\cdots\otimes V}_{r}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{s}$$

Ex: An inner product $\langle\cdot,\cdot\rangle$ on V is a symmetric $(0,2)$-tensor on V:

$\langle\cdot,\cdot\rangle$ defines a bilinear map $V\times V\to\mathbb{R}$. By the universal property, this induces a ! linear map $V\otimes V\to\mathbb{R}$.

So $\langle\cdot,\cdot\rangle$ uniquely corresponds to an element of $(V\otimes V)^* = V^*\otimes V^*$

The direct sum $\mathcal{T}(V) = \bigoplus_{r,s\geq 0} V_r^s$ where $V_0^0 =$ ground field of V is the tensor algebra of V.

$\mathcal{T}(V)$ is noncommutative ($v_1\otimes v_2 \neq v_2\otimes v_1$), associative & (bi-)graded.

Now, define the tensor bundle of type (r,s) over a mfld M to be

$$\mathcal{T}_r^s(M) = \bigcup_{p\in M} (T_pM)_r^s$$

which is a smooth mfld. A tensor field of type (r,s) on M is a smooth section of the (r,s)-tensor bundle; that is, a smooth map $M\to\mathcal{T}_r^s(M)$

Ex:
- A vector field on M is a $(1,0)$-tensor field on M.
- A Riemannian metric on M is a symmetric $(0,2)$-tensor field.
- A 1-form is a $(0,1)$-tensor field.