Math 670: Day 7

Recall from last time:

Prop: Let X & Y be smooth vector fields on M, let $p \in M$ & let φ_t be the local flow of X in a nbhd of p. Then

$$[X,Y]_p = \lim_{t \to 0} \frac{Y_{\varphi_t(p)} - d\varphi_t(Y_p)}{t}$$

Differential Forms

"Def": A (smooth) differential k-form on a mfd M is a C^∞, alternating, multilinear map

$$\omega: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k \text{ copies}} \to C^\infty(M)$$

where $\mathfrak{X}(M)$ is the $C^\infty(M)$ module of smooth vector fields on M.

(i.e., $\omega(X_1, .., X_{i-1}, fX + gY, X_{i+1}, .., X_k) = f\,\omega(X_1, .., X_{i-1}, X, X_{i+1}, .., X_k) + g\,\omega(X_1, .., X_{i-1}, Y, X_{i+1}, .., X_k)$

& $\omega(X_1, \dots, X, .., X, .., X_k) = 0$)

The vector space of k-forms is denoted $\Omega^k(M)$.

This def. is kind of vague & useless, so let's see a concrete ex.

Ex: Let M be a mfd & $X \in \mathfrak{X}(M)$ a vector field. Assume M has a Riemannian metric g, meaning that $\forall p \in M$ & $u, v \in T_p(M)$, $g_p(u,v)$ gives an inner product on $T_p(M)$.

Then X corresponds to a unique 1-form α defined by

$$\alpha_p(Y) = g_p(X,Y).$$

Then $\alpha: \mathfrak{X}(M) \to C^\infty(M)$ is smooth since the Riemannian metric is.

It's trivially alternating

It's linear since g is.

OTOH, any 1-form β is just a linear functional on each T_pM, so $\beta_p \in (T_pM)^*$ $\forall\, p \in M$. But then

$\exists\, u_p \in T_pM$ s.t. $\beta_p(v) = g_p(u_p, v)$ $\forall\, v \in T_pM$ (this is the Riesz Representation Thm). Then the smoothness of β allows us to define a smooth vector field U by $U_p = u_p$.

In other words, g determines an isomorphism $\mathfrak{X}(M) \to \Omega^1(M)$ by $X \mapsto g(X, \cdot)$.

Also, 0-forms are clearly just C^∞ functions on M, so $\Omega^0(M) = C^\infty(M)$.

2 questions: ① What about $\Omega^k(M)$ for other k?

② What are the gradient & divergence (& curl in 3 dimensions) in this language?

For ②, we know the gradient of a function is a v.f., given in local coords by

$$\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

so the corresponding 1-form is α given by

$$\alpha_p(Y) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} b_i = \left(\sum b_i \frac{\partial}{\partial x_i}\right) f = (Yf)(p) = df_p(Y)$$

where $Y = \sum b_i \frac{\partial}{\partial x_i}$ in local coords. In other words, the 1-form identified w/ ∇f is exactly the differential df.

(In general, note that differentials are 1-forms: given $f \in C^\infty(M)$, $df_p \in (T_pM)^*$, so $df \in T^*M \cong \Omega^1(M)$)

i.e. $\nabla : C^\infty(M) \to \mathfrak{X}(M)$ corresponds to the exterior derivative d!

$$\begin{tikzcd} \nabla : C^\infty(M) \arrow[r] \arrow[d, equal] & \mathfrak{X}(M) \arrow[d, "\cong"] \\ d : \Omega^0(M) \arrow[r] & \Omega^1(M) \end{tikzcd}$$

$$f \mapsto df$$

Later we'll formulate an invariant definition of d.

$$\begin{tikzcd} \mathfrak{X}(M) \arrow[r, "\nabla\cdot"] \arrow[d, "\cong"] & C^\infty(M) \arrow[d, "\cong"] \\ d : \Omega^{n-1}(M) \arrow[r] & \Omega^n(M) \end{tikzcd}$$

It turns out that the divergence is also an exterior derivative, this time $d : \Omega^{n-1}(M) \to \Omega^n(M)$

An n-form on M^n is an alternating multilinear map $\omega : (\mathfrak{X}(M))^n \to C^\infty(M)$. At each $p \in M$, $T_pM \cong \mathbb{R}^n$, so ω_p is an alternating multilinear map $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n} \to \mathbb{R}$. Up to a scalar, the unique such thing is the determinant (to make this rigorous, need a Riemannian metric), so each ω_p is a scalar multiple of the determinant, meaning that $\omega = f\, dvol_M$, where $dvol_M$ is the name we give to the n-form which is just the determinant on each T_pM. Therefore, $\Omega^n(M) \cong C^\infty(M)$.