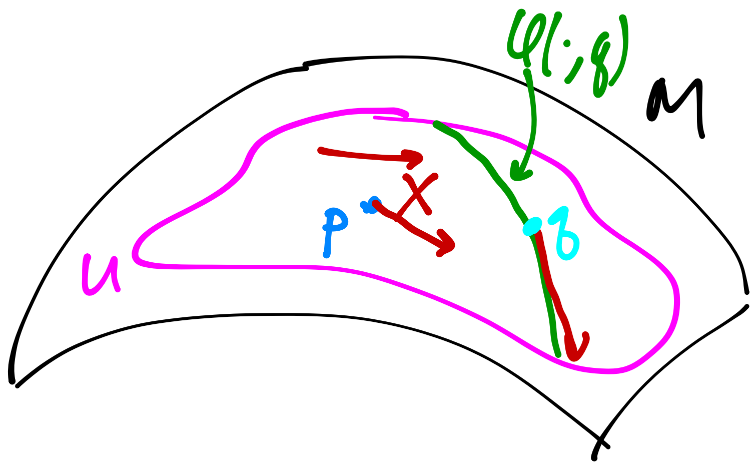
Math 670: Day 6

Def: Let X be a v.f. on M. A curve $\alpha:(a,b)\to M$ is called an *integral curve* (or *trajectory*) for X if $\alpha'(t)=X_{\alpha(t)}$ $\forall\, t\in(a,b)$.

Prop: Let X be a v.f. on M & let $p\in M$. Then $\exists$ nbhd U of p, $\delta>0$, & map $\varphi:(-\delta,\delta)\times U\to M$ s.t. $t\mapsto\varphi(t,q)$ is the unique smooth curve satisfying $\frac{\partial\varphi}{\partial t}=X_{\varphi(t,q)}$ & $\varphi(0,q)=q$.

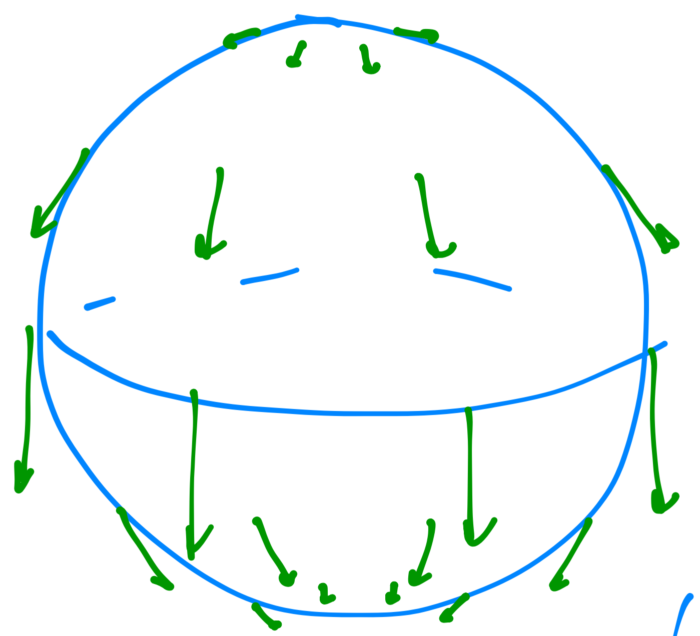
Proof: Since M is locally diffeo. to $\mathbb{R}^n$, this follows from the fundamental theorem on existence & uniqueness of solutions to ODEs. ∎



Often denote $\varphi_t(q)=\varphi(t,q)$ & call it the *local flow* of X.

Ex: $M=S^2$, $X_{(x,y,z)}=(zx,zy,z^2-1)$

Then the local flow pushes the mass of the sphere down towards the south pole.



Note: This is the image under the differential of inverse stereographic projection of the v.f. $(-x,-y)$ on $\mathbb{R}^2$.

This is an example of a *complete* v.f. since the local flow exists for all time at all points.

If the mfld were instead the sphere w/ the antarctic circle removed, the v.f. would not be complete.

Now, we can interpret the Lie bracket $[X,Y]$ as a sort of derivative of Y along the trajectories of X.

Prop: Let X & Y be smooth vector fields on M, let $p\in M$ & let φ_t be the local flow of X in a nbhd of p. Then

$$[X,Y]_p=\lim_{t\to 0}\frac{Y_{\varphi_t(p)}-d\varphi_t(Y_p)}{t}$$

Proof: For f a differentiable function in a nbhd of p, the goal is to show that

$$\lim_{t\to 0}\frac{Y_{\varphi_t(p)}-d\varphi_t(Y_p)}{t}=((XY-YX)f)(p).$$

Notice that, by definition of the differential, $((d\varphi_t Y)f)(\varphi_t(p))=Y(f\circ\varphi_t)(p)$, so the LHS is

$$\lim_{t\to 0}\frac{(Yf)(\varphi_t(p))-Y(f\circ\varphi_t)(p)}{t}$$

If we could get $\lim_{t\to 0} \frac{(Yf)(\varphi_t(p)) - (Yf)(p)}{t} = \frac{d(Yf\circ\varphi_t)}{dt}\Big|_{t=0} = X(Yf)(p)$ to show up, we would be in business.

To that end, define $h(t,q) := f(\varphi_t(q)) - f(q)$ $\forall q$ in the domain of f & all t sufficiently small.

Claim: $\exists$ smooth function g s.t. $t\, g(t,q) = h(t,q)$; in particular, $g(0,q) = \frac{\partial h(t,q)}{\partial t}\Big|_{t=0}$

Assuming the claim is true, we have that

$$(f\circ\varphi_t)(q) = f(q) + t\,g(t,q) \quad \& \quad g(0,q) = Xf(q).$$

But then

$$((d\varphi_t Y) f)(\varphi_t(p)) = (Y(f\circ\varphi_t))(p) = (Yf)(p) + t(Y g(t,\cdot))(p)$$

&

$$\lim_{t\to 0} \frac{(Yf)(\varphi_t(p)) - Y(f\circ\varphi_t)(p)}{t} = \lim_{t\to 0} \frac{(Yf)(\varphi_t(p)) - ((Yf)(p) + t\,(Yg(t,\cdot))(p))}{t}$$

$$= \lim_{t\to 0} \frac{(Yf)(\varphi_t(p)) - (Yf)(p)}{t} - (Y g(0,\cdot))(p)$$

$$= (X(Yf))(p) - (Y(Xf))(p)$$

$$= ([X,Y]f)(p),$$

as desired. $\blacksquare$

Proof of Claim: Define $g(t,q) = \int_0^1 \frac{\partial h(ts,q)}{\partial (ts)}\, ds$. Then letting $u = ts$, $du = t\,ds$, so

$$g(t,q) = \frac{1}{t}\int_0^1 \frac{\partial h(ts,q)}{\partial (ts)}\, t\,ds = \frac{1}{t}\int_0^t \frac{\partial h(u,q)}{\partial u}\, du = \frac{1}{t} h(t,q). \quad \blacksquare$$