Math 670: Day 5

Vector Fields:

Df: A *vector field* X on a smooth mfd M is a smooth section of the tangent bundle TM; i.e., a smooth map $X: M \to TM$. Given how we've defined tangent vectors, this means a vector field is just a linear derivation on $C^\infty(M)$.

Note: If $\varphi: U \subseteq \mathbb{R}^n \to M$ is a coord. chart at $p \in M$, then

$$X(p) = \sum a_i(p) \frac{\partial}{\partial x_i},$$ where each $a_i : U \to \mathbb{R}$ is smooth & $\{\frac{\partial}{\partial x_i}\}$ is the basis of T_pM associated w/ (U,φ).

- As a map $C^\infty(M) \to C^\infty(M)$, a vector field acts as

$$(Xf)(p) = \sum a_i(p) \frac{\partial f}{\partial x_i}\Big|_p$$

which notice really is a smooth function on M.

Df: If X & Y are smooth vector fields on M, the *Lie bracket* of X & Y is a vector field $[X,Y]$ defined by

$$[X,Y](f) := X(Yf) - Y(Xf)$$

In coords., if $X = \sum a_i \frac{\partial}{\partial x_i}$ & $Y = \sum b_i \frac{\partial}{\partial x_i}$, then

$$[X,Y]f = X(Yf) - Y(Xf) = X\left(\sum b_i \frac{\partial f}{\partial x_i}\right) - Y\left(\sum a_i \frac{\partial f}{\partial x_i}\right) = \sum_{i,j} a_j\left(\frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} + b_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right) - \sum_{i,j} b_j\left(\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} - a_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right)$$

$$= \sum \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i} + \sum a_i b_j \left(\cancel{\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}}\right) \to 0$$

$$= \underbrace{\left(\sum_{i,j} a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}\Big)}_{\text{local coord expression for } [X,Y]} f$$

Prop: If X, Y, Z are smooth vector fields on M & $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$, then

① $[X,Y] = -[Y,X]$ (anticommutivity)

② $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ (linearity)

③ $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity)

④ $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

$\blacksquare$

Proof: Exercise.

Ex: Consider $SO(n)$. Then an element of $T_I SO(n)$ is a tangent vector; i.e., the tangent to a curve through I. Let

$\alpha : (-\varepsilon, \varepsilon) \to SO(n)$ s.t. $\alpha(0) = I$. Then $\alpha(t)$ is an orthogonal matrix $\forall\, t \in (-\varepsilon, \varepsilon)$, so $\alpha(t)\alpha(t)^T = I$.

Differentiate: $\alpha'(t)\alpha(t)^T + \alpha(t)\alpha'(t)^T = 0$. At $t=0$, this reduces to $\alpha'(0) + \alpha'(0)^T = 0$, so $\alpha'(0) = -\alpha'(0)^T$, so

$\alpha'(0)$ is skew-symmetric. The converse is also true, so we can identify $T_I SO(n)$ w/ the set of skew-symmetric $n \times n$ matrices.

Also, note that if $\Delta \in T_I SO(n)$, then $e^{\Delta} \in SO(n)$ b/c $(e^{\Delta})^T = e^{(\Delta^T)} = e^{-\Delta} = (e^{\Delta})^{-1}$ & $\det(e^{\Delta}) = e^{\operatorname{tr}\Delta} = e^0 = 1$.

You can prove that this map is actually surjective as well.

Now, if $Q \in SO(n)$ & we define $l_Q : SO(n) \to SO(n)$ by $l_Q(A) = QA$, then

$$(dl_Q)_I\, \Delta = Q\Delta \qquad \text{(check this if you don't believe it!)}$$

Since $(dl_Q)_I : T_I SO(n) \to T_I SO(n)$ is full rank it's surjective, so $T_Q SO(n)$ consists of matrices of the form $Q\Delta$ for Δ skew-symmetric.

Therefore, a vector field X on $SO(n)$ is given at each pt. $Q \in SO(n)$ by $X_Q = Q\Delta_Q$.

If $\Delta_Q = \Delta_I$ $\forall\, Q \in SO(n)$ is called left-invariant, & the space of left-inv't vf's on $SO(n)$ can then be identified w/ $T_I SO(n)$.

Now, the Lie bracket on left-inv't vf's is especially nice: if X & Y are left-inv't vf's w/ $X_I = \Delta_X$ & $Y_I = \Delta_Y$, then $[X,Y]_I = \Delta_1\Delta_2 - \Delta_2\Delta_1$ & $[X,Y]_Q = Q(\Delta_1\Delta_2 - \Delta_2\Delta_1)$, so the Lie bracket becomes a binary operation on $T_I SO(n)$. This makes $T_I SO(n)$ a Lie algebra denoted $\mathfrak{so}(n)$.
