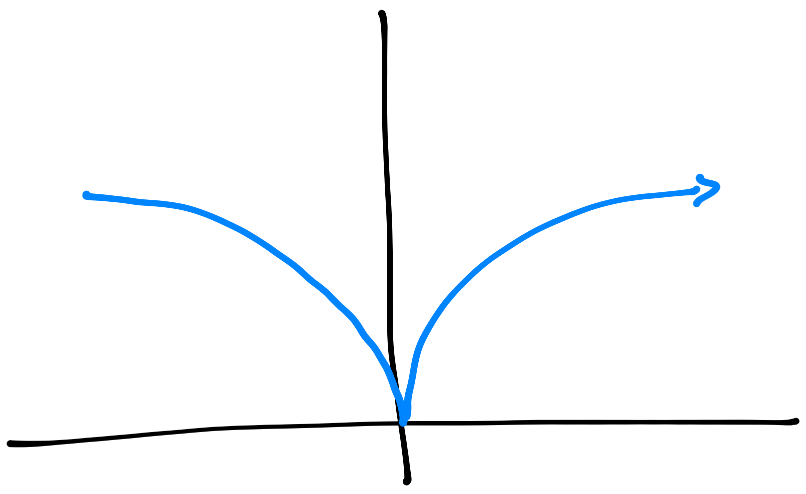
Math 670: Day 4

Def: Suppose $f: M \to N$ is smooth. Then f is an *immersion* if df_p is injective for all $p \in M$. If f is also a homeomorphism (continuous bijection w/ continuous inverse), then f is an *embedding*. The image of an embedding is a *submanifold*. If f is bijective & f^{-1} is smooth, then f is a *diffeomorphism*. f is called a *local diffeomorphism* at $p \in M$ if $\exists$ nbhd U of p & V of $f(p)$ s.t. $f: U \to V$ is a diffeomorphism.

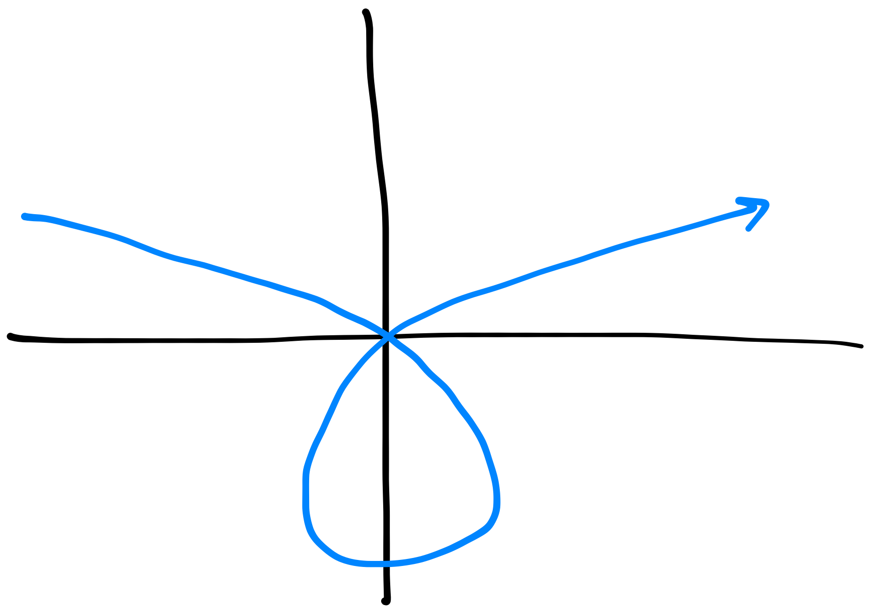
If df_p is surjective then f is a *submersion*.

Exs: • $\alpha: \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^2, t^3)$.



Not an immersion b/c $d\alpha_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has rank 0

• $\alpha(t) = (t^3 - 4t,\ t^2 - 4)$



Is an immersion since $d\alpha_t = \begin{pmatrix} 3t^2 - 4 \\ 2t - 4 \end{pmatrix}$ always has rank 1.

But clearly not an embedding.

Prop: Suppose $f: M \to N$ is smooth. If $p \in M$ & $df_p : T_pM \to T_{f(p)}N$ is an isomorphism, then f is a local diffeo. at p.

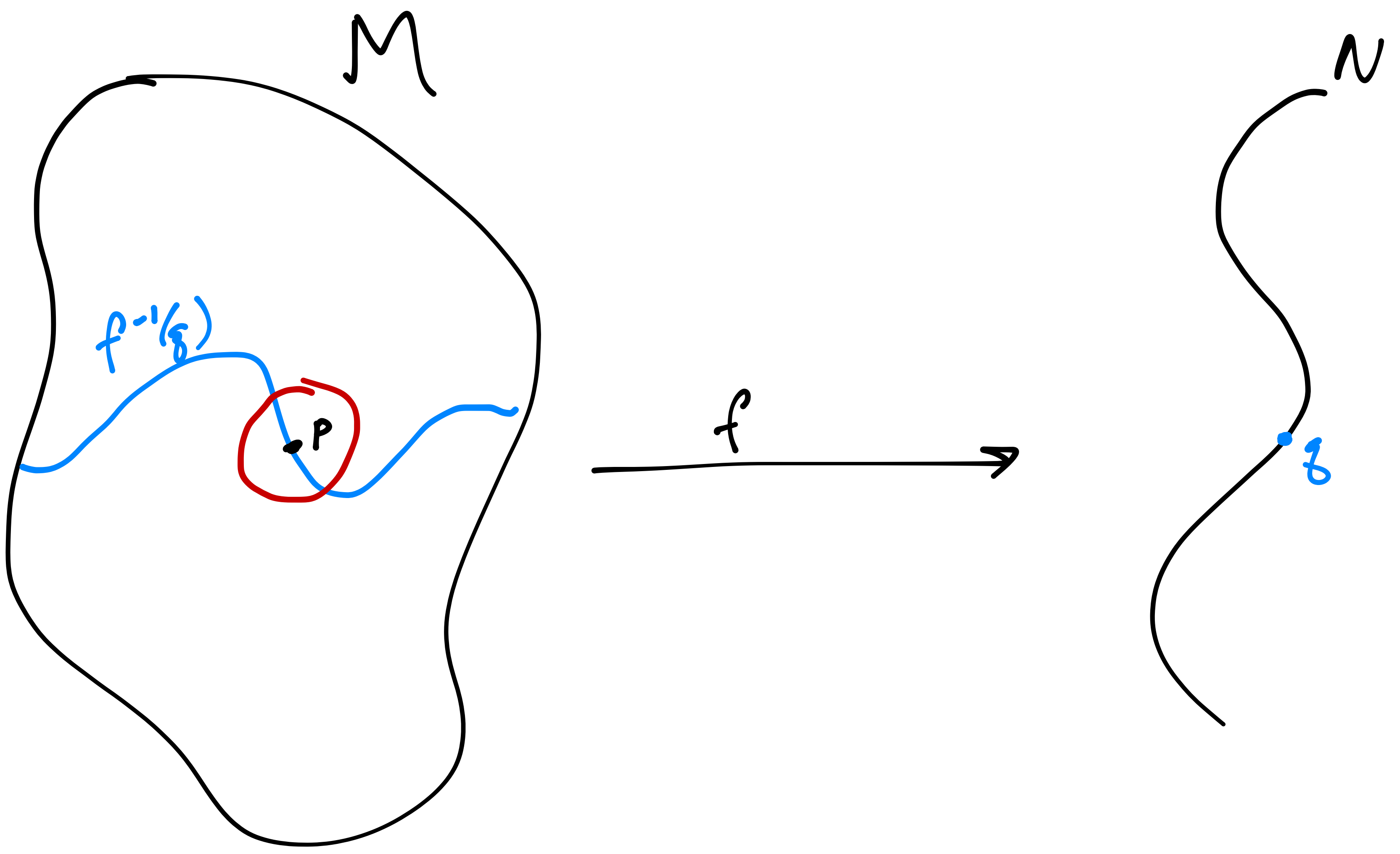
Proof: Inverse function theorem applied to $\psi^{-1} \circ f \circ \varphi : U \to V$, where (U, φ) is a local coord. chart at p & (V, ψ) is a local coord chart at $f(p)$.

Recall:

$$\boxed{\text{IFT: If } F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n \text{ is given by } (x_1, \dots, x_n) \mapsto (y_1, \dots, y_n), \text{ then } dF = \left(\frac{\partial y_i}{\partial x_j}\right) \text{ being nonsingular at } p \in U \text{ implies that } F \text{ is a local diffeo. at } p.}$$

Def: Let $f: M^m \to N^n$ be smooth. A point $p \in M$ is a *critical point* of f if $df_p : T_pM \to T_{f(p)}N$ is not surjective; then $f(p)$ is a *critical value* of f. A point $q \in N$ which is not a critical value is a *regular value*.

Thm: If $q \in N$ is regular, then $f^{-1}(q)$ is a smooth submanifold of M of dimension $m-n$.



Ex: Define $f: \mathbb{R}^n \to \mathbb{R}$ by $f(x_1,\dots,x_n) = \sum x_i^2$. Then all $r \neq 0$ are regular values, so the sphere $f^{-1}(r)$ of radius $\sqrt{r}$ in $\mathbb{R}^n$ is a smooth submanifold of codimension 1.

· Define $f: GL(n,\mathbb{R}) \to \{\text{symmetric } n\times n \text{ matrices}\}$ by $f(A) = AA^T$. Then f is a submersion & $f^{-1}(I) = O(n)$ is a smooth submanifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Proof of Thm: Suppose (U,φ) is a coord. chart centered at $p \in f^{-1}(q)$ & (V,ψ) a chart at q. Then $g = \psi^{-1} \circ f \circ \varphi : U \to \mathbb{R}^n$ and, by assumption, $dg_{\vec{0}}$ is surjective. Hence, after a linear change of coords $dg_{\vec{0}}$ can be written as the $n \times m$ matrix $\begin{bmatrix} I_n & O \end{bmatrix}$. Then define $G(x_1,\dots,x_m) = (g(x_1,\dots,x_n), x_{n+1},\dots,x_m)$, which has differential $dG_{\vec{0}} = I_m$. By IFT, G is a local diffeo. at $\vec{0}$ & $g \circ G^{-1}$ is the standard projection onto the first n coordinates. Then in these coords $f^{-1}(q) \cap U = \varphi(0,0,\dots,0,x_{n+1},\dots,x_m)$, so $x_{n+1},\dots,x_m$ give local coords near p for $f^{-1}(q)$. Of course we can do the same at any other point of $f^{-1}(q)$. $\blacksquare$