Math 670: Day 38

Suppose we have a circle action on a symplectic mfd (M,ω) that preserves the symplectic structure; i.e., for $\theta \in S^1$, $f_\theta : M \to M$ is a diffeo. & $f_\theta^* \omega = \omega$.

If the orbits of the action are integral curves of a Hamiltonian v.f. X_H, meaning $X_H = \frac{df_\theta}{d\theta}$ then we know the Hamiltonian $H : M \to \mathbb{R}$ is constant along the orbits of the action & we have $dH = \iota_{X_H}\omega$.

Ex: Rotating S^2 around the z-axis. Then the Hamiltonian is $H(\theta, z) = z$.

This generalizes to arbitrary group actions: Suppose G is a Lie group acting on (M,ω) by symplectomorphisms; i.e. $\psi_g : M \to M$ is a symplecto. for each $g \in G$. Let $\mathfrak{g}$ be the Lie alg. of G & let $\mathfrak{g}^*$ be its dual. Then the action is Hamiltonian if $\exists$ a map

$$\mu : M \to \mathfrak{g}^* \quad \text{s.t.}$$

① For each $X \in \mathfrak{g}$, let $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X\rangle$ be the component of μ along X, and let $X^\# \in \mathfrak{X}(M)$ be the v.f. generated by the one-parameter subgp $\{\exp tX : t \in \mathbb{R}\} \subseteq G$. Then $d\mu^X = \iota_{X^\#}\omega$; i.e., μ^X is a Hamiltonian for $X^\#$.

② μ is equivariant w.r.t. the action of G on M & the coadjoint action Ad^* of G on $\mathfrak{g}^*$:

$$\mu \circ \psi_g = Ad_g^* \circ \mu \quad \forall g \in G.$$

The map μ is called the moment map (or momentum map) associated to the action of G on M.

When $G = (S^1)^n$ acts Hamiltonianly on (M^{2n}, ω), then (M,ω) is called a toric symplectic mfd.

Thm (Atiyah & Guillemin-Sternberg): The image of the moment map is a convex polytope whose vertices are the images of fixed points of the action.

Thm (Delzant): Toric symplectic mfds are classified by the moment polytope.

Thm (Duistermaat-Heckman) If $P \subseteq \mathbb{R}^n \cong (\mathfrak{u}(1)^n)^*$ is the moment polytope of a toric mfd (M,ω), then $\mu : M \to P$ pushes forward the symplectic (or Liouville) measure on M forward to Lebesgue measure on P. In other words, from a measure theory perspective M & $P \times (S^1)^n$ are indistinguishable!

Thm (Marsden-Weinstein & Meyer): Suppose G is a compact Lie group with a Hamiltonian action on (M,ω). Let $i : \mu^{-1}(\vec{0}) \to M$ be the inclusion and assume G acts freely on $\mu^{-1}(\vec{0})$. Then

① The symplectic reduction $M/\!\!/_{\vec{0}}\, G := \mu^{-1}(\vec{0})/G$ is a manifold.

② $\pi : \mu^{-1}(0) \to M/\!\!/_0\, G$ is a principal G-bundle

③ There is a symplectic form $\overline{\omega}$ on $M/\!\!/_0\, G$ s.t. $i^*\omega = \pi^*\overline{\omega}$.

Key Example: Let $SO(3)$ act on the obvious way on $(S^2)^n$ w/ the standard symplectic form. This action is obviously by symplectomorphisms since it is area-preserving on each factor. In fact, it is Hamiltonian: $\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} \right\} \cong (\mathbb{R}^3, \times)$ & we've seen that, for example, if $A = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} \cos(ta) & -\sin(ta) & 0 \\ \sin(ta) & \cos(ta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so $A^\#$ is just $\frac{\partial}{\partial \theta}$ on each factor S^2 & the corresponding moment map is $H_A((x_1,y_1,z_1), \dots, (x_n,y_n,z_n)) = \sum z_i$ (since the symplectic form ω is the sum $\sum \omega_i$).

Likewise, $H_B = \sum y_i$ & $H_C = \sum x_i$, so the moment map is $\mu : (S^2)^n \to \mathbb{R}^3$ given by (after permuting coords)

$$((x_1,y_1,z_1), \dots, (x_n,y_n,z_n)) \mapsto \begin{pmatrix} \sum x_i \\ \sum y_i \\ \sum z_i \end{pmatrix}$$

In other words, if we think of $(S^2)^n$ as the space of equilateral polygonal arms in $\mathbb{R}^3$ up to translation, then $\mu^{-1}(0)$ is the space of equilateral closed polygons in $\mathbb{R}^3$ up to translation.

Thus, doing symplectic reduction, the moduli space of equilateral polygons up to translation & rotation is $\mu^{-1}(0)/SO(3) = (S^2)^n /\!/_{\vec{0}}\, SO(3)$, which has a natural symplectic structure. Even better, it's toric, so up to measure it looks like the product of a convex polytope & a torus!

(Cantarella-Shonkwiler & Cantarella-Shonkwiler-Uehara): Various sampling algorithms.