Math 6170: Day 37

Equilateral polygons & connections to symplectic geometry

Suppose instead we are interested in understanding equilateral polygons in $\mathbb{R}^3$ (since this is the most common polymer model)

Again treating the edges as vectors, we want points $(\vec{e}_1, \ldots, \vec{e}_n) \in (S^2)^n$ s.t. $\sum \vec{e}_i = \vec{0}$. Since this equation is really 3 equations (1 for each coordinate), we're talking about a $(2n-3)$-dim'l space or, after modding out by $SO(3)$, a $(2n-6)$-dim'l space.

Up in the quaternionic world, we'll see a product of 3-spheres: $S^3(1) \times \ldots \times S^3(1)$, but this has not turned out to be a particularly productive approach.

Instead, it turns out to be much better to use the symplectic geometry of $(S^2)^n$.

Def: A symplectic structure on a $2n$-dim'l mfd is a closed, nondegenerate 2-form $\omega \in \Omega^2(M)$.

Here non-degenerate means that if $\exists\, X \in T_pM$ s.t. $\omega_p(X,Y) = 0$ $\forall\, Y \in T_pM$, then $X = 0$ or, equivalently, that

$$\omega^{\wedge n} = \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}} \text{ is nowhere-vanishing.}$$

Ex: $\mathbb{C}^n$ w/ $\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$.

Ex: Any oriented surface. The area form is the symplectic form.

e.g., S^2 w/ area form ω given by $\omega_p(u,v) = \langle p, u \times v \rangle$ for all $u, v \in T_pS^2$.

Def: Let $(M_1, \omega_1), (M_2, \omega_2)$ be symplectic $2n$-mfds & let $\varphi: M_1 \to M_2$ be a diffeomorphism. φ is a symplectomorphism if $\varphi^*\omega_2 = \omega_1$.

Thm (Darboux): Let (M^{2n}, ω) be symplectic & let $p \in M$. Then $\exists$ coord. chart $\varphi: U \to M$, $U \subseteq \mathbb{R}^{2n}$, centered at p s.t.

$$\varphi^*\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

In other words, there is no interesting local symplectic geometry.

Hamiltonians

Let (M,ω) be symplectic & suppose $H: M \to \mathbb{R}$ is smooth. Then $dH \in \Omega^1(M)$; by nondegeneracy, $\exists!\, X_H \in \mathfrak{X}(M)$ s.t.

$$\iota_{X_H}\omega = \omega(X_H,\cdot) = dH.$$

Let $\varphi_t : M \to M$ be the 1-parameter family of diffeos generated by X_H; i.e., $\varphi_0 = id_M$ & $\frac{d\varphi_t}{dt}\circ\varphi_t^{-1} = X_H$

Lemma: $\varphi_t^*\omega = \omega$, so H induces a family of symplectomorphisms.

Proof:
$$\frac{d}{dt}\varphi_t^*\omega \underset{\substack{\text{Def of Lie deriv.}\\ \text{(see Day 13)}}}{=} L_{X_H}\omega \underset{\substack{\text{Cartan's}\\ \text{Magic}\\ \text{Formula}}}{=} d\underbrace{\iota_{X_H}\omega}_{dH} + \iota_{X_H}\underbrace{d\omega}_{0} = 0. \qquad \blacksquare$$

The vector field X_H is called a **Hamiltonian vector field** with **Hamiltonian function** H.

Important Example: Consider $(S^2, d\theta\wedge dh)$ & let $H: S^2\to\mathbb{R}$ be the height function; i.e., $H(\theta,h)=h$. Then

$$dH = dh = \iota_{X_h}(d\theta\wedge dh) = d\theta\wedge dh(X_H,\cdot) \iff X_H = \frac{\partial}{\partial\theta},$$

so the flow is $\varphi_t(\theta,h) = (\theta+t,h)$, rotation around the vertical axis, which of course preserves H.

In fact, the local flow of any Hamiltonian vector field **always** conserves the associated Hamiltonian (in other words, integral curves are contained in level sets):

$$\frac{d}{dt}\left(H\circ\varphi_t(x)\right) = dH(X_H) = \iota_{X_H}\omega(X_H) = \omega(X_H,X_H) = 0.$$

Ex (Hamiltonian Mechanics): Consider $\mathbb{R}^{2n}$ w/ coords. $(q_1,\dots,q_n,p_1,\dots,p_n)$ & $\omega = \sum dq_j\wedge dp_j$. Then for any Hamiltonian v.f. X_H,

$\varphi_t = (q(t),p(t))$ is an integral curve $\iff$ $\begin{cases} \frac{dq_i}{dt}(t) = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt}(t) = -\frac{\partial H}{\partial q_i}\end{cases}$ (Hamilton equations)

Indeed, if $X_H = \sum\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right)$, then

$$\iota_{X_H}\omega = \sum \iota_{X_H}(dq_j\wedge dp_j) = \sum\left[(\iota_{X_H}dq_j)\wedge dp_j - dq_j\wedge(\iota_{X_H}dp_j)\right] = \sum\left(\frac{\partial H}{\partial p_j}dp_j + \frac{\partial H}{\partial q_j}dq_j\right) = dH$$

Now, consider $n=3$ & a particle of mass m moving under a potential $V(q)$ along a curve $q(t)$ s.t.

$$m\frac{d^2q}{dt^2} = -\nabla V(q).$$

Introduce momenta $p_i = m\frac{dq_i}{dt}$ & the energy function $H(p,q) = \frac{1}{2m}|p|^2 + V(q)$. Let $\mathbb{R}^6 = T^*\mathbb{R}^3$ be the corresponding phase space. Then

Newton's second law is equivalent to the Hamilton equations:

$$\begin{cases} \frac{dq_i}{dt} = \frac{1}{m}p_i = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = m\frac{d^2q_i}{dt^2} = -\frac{\partial V}{\partial q_i} = -\frac{\partial H}{\partial q_i}\end{cases}$$

and so we see that the energy H is conserved by the physical motion.