Math 670: Day 36

And so now the question becomes: what is the preimage of the closed framed polygons?

Of course, we know how to answer this question. If $q = a + bj$ for $a, b \in \mathbb{C}$, then

$$q \mapsto \bar{q}\, i\, q = i(|a|^2 - |b|^2 + 2\bar{a}bj)$$

And hence $\mathrm{FrHopf}(q_1, \ldots, q_n)$ is closed $\Leftrightarrow 0 = \sum \bar{q}_\ell\, i\, q_\ell = \sum i(|a_\ell|^2 - |b_\ell|^2 + 2\bar{a}_\ell b_\ell j)$

$$= i(\|\vec{a}\|^2 - \|\vec{b}\|^2 + 2\langle \vec{a}, \vec{b}\rangle j)$$

where $\vec{q} = \vec{a} + \vec{b}j$ for $\vec{a}, \vec{b} \in \mathbb{C}^n$ & $\langle \cdot, \cdot \rangle$ is the Hermitian dot product. This proves

Prop (Hausmann-Knutson, 1997): $\mathrm{FrHopf}(\vec{q})$ is closed $\Leftrightarrow$ $\vec{a}, \vec{b}$ are the same length & (Hermitian) orthogonal.

(Should also mention Howard-Manon-Millson, 2011 for this, esp. the "unploded, spin-framed" part).

And finally:

Prop (Hausmann-Knutson, 1997): The total perimeter of $\mathrm{FrHopf}(\vec{q})$ is $\|\vec{a}\|^2 + \|\vec{b}\|^2$.

Proof: Total perimeter $= \sum \|\bar{q}_\ell\, i\, q_\ell\| = \sum \|q_\ell\|^2 = \sum (|a_\ell|^2 + |b_\ell|^2) = \|\vec{a}\|^2 + \|\vec{b}\|^2$. ▨

(Mathematica demo)