Math 670: Day 35

Having dealt w/ polygons in the plane, the obvious question is: what is the corresponding story for polygons in $\mathbb{R}^3$?

The key question is how to generalize the squaring map that took us from $\mathbb{C}^n$ to $\mathbb{C}^n$.

There are two more or less equivalent ways to think about generalizing the squaring map to higher dimensions.

① The squaring map $S^1 \to S^1$ is a low-dim'l version of the Hopf fibration. Specifically, we have the fibration

$$\begin{tikzcd} S^0 \arrow[r, hook] & S^1 \arrow[d] \\ & S^1 \end{tikzcd}$$

where the vertical map is the squaring map, which we can think of as follows:

the base $S^1 \cong \mathbb{RP}^1$, & the map sends points on the unit circle to the (real) line through the origin containing them.

Of course, this generalizes to a map $\begin{tikzcd} S^0 \arrow[r, hook] & \mathbb{C}^* \arrow[d] \\ & \mathbb{C}^* \end{tikzcd}$ &, if we drop the requirement that it be a fibration, a map $\begin{tikzcd} \mathbb{C} \arrow[d] \\ \mathbb{C} \end{tikzcd}$

Now, the real Hopf map is the analogous thing for cplx lines in $\mathbb{C}^2$ (rather than real lines in $\mathbb{R}^2$): we get a fibration

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d] \\ & S^2 \cong \mathbb{CP}^1 \end{tikzcd}$$

where pts on S^3 get mapped to the cplx line containing them. In coords., if $(z_1, z_2) \in \mathbb{C}^2$ we can write the quaternion $p = z_1 + z_2 j$. Then the intersection of the cplx line containing (z_1, z_2) w/ S^3 is the circle $\{(e^{i\theta} z_1, e^{i\theta} z_2) : \theta \in \mathbb{R}\}$

Now, $\bar{p} i p = (\bar{z}_1 - z_2 j) i (z_1 + z_2 j) = i \bar{z}_1 z_1 + \bar{z}_1 i z_2 j - z_2 j i z_1 - z_2 j i z_2 j$

$$\begin{aligned}
&= i \bar{z}_1 z_1 + i \bar{z}_1 z_2 j + z_2 i j z_1 + z_2 i j z_2 j \\
&= i \bar{z}_1 z_1 + i \bar{z}_1 z_2 j + i z_2 \bar{z}_1 j + i z_2 \bar{z}_2 j^2 \\
&= i\left(\bar{z}_1 z_1 - \bar{z}_2 z_2 + 2 \bar{z}_1 z_2 j\right) \\
&= i\left(|z_1|^2 - |z_2|^2 + 2 \bar{z}_1 z_2 j\right).
\end{aligned}$$

& $\overline{(e^{i\theta} p)}\, i\, (e^{i\theta} p) = i\left(|e^{i\theta} z_1|^2 - |e^{i\theta} z_2|^2 + 2 e^{-i\theta} \bar{z}_1 e^{i\theta} z_2 j\right) = i\left(|z_1|^2 - |z_2|^2 + 2 \bar{z}_1 z_2 j\right) \ldots$

so the map $p \mapsto \bar{p} i p$ sends the entire cplx line to a single point.

Of course, this can also be generalized to $\begin{tikzcd} S^1 \arrow[r, hook] & \mathbb{H}^* \arrow[d] \\ & \mathbb{R}^3 \setminus \{\vec{0}\} \end{tikzcd}$ or, dropping the fibration-ness of it, $\mathrm{Hopf} : \mathbb{H} \to \mathbb{R}^3$.

② We can view the squaring map $\mathbb{Z}/2 \hookrightarrow S^1 \to S^1$ as the SES of Lie groups $1 \to \mathbb{Z}/2 \to Spin(2) \to SO(2) \to 1$

Here $Spin(2) = U(1)$ acts on $\mathbb{R}^2 = \mathbb{C}$ by "double phase rotation": $u(z) = u^2 z$.

The corresponding thing one dimension up is $1 \to \mathbb{Z}/2 \to Spin(3) \to SO(3) \to 1$, where $Spin(3) = SU(2) = Sp(1) = S^3$ &

the map $SU(2) \to SO(3)$ is essentially the map you're exploring on HW.

In fact, to accord w/ existing literature, I use a slightly different map than in HW, namely the map

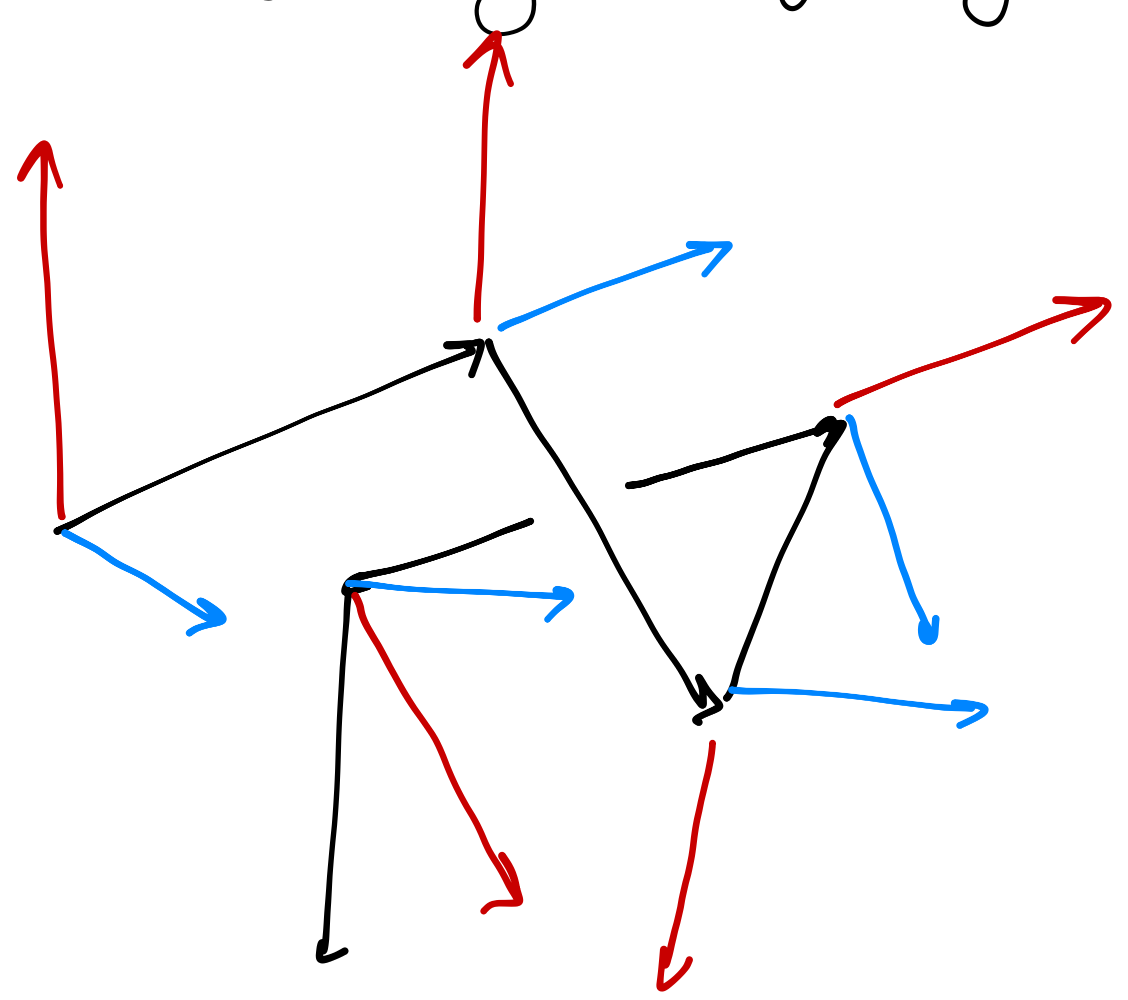
$$\begin{aligned} S^3 &\longrightarrow SO(3) \\ p &\longmapsto \begin{bmatrix} \bar{p} i p & \bar{p} j p & \bar{p} k p \end{bmatrix} \end{aligned}$$

Notice that the first column is the same map from ①.

Again, we can generalize to a map $\mathbb{H} \to CSO(3)$ (the cone over $SO(3)$; i.e., the set of all 3x3 matrices w/ pairwise orthogonal columns which all have the same norm [which could be zero!]).

In fact, we see from ② that the natural base object is the space of framed polygons in $\mathbb{R}^3$:

each edge is actually a triple of mutually perpendicular vectors of a common length (a sort of discrete version of a framed curve)



So if we let FArm(n) be the set of framed polygonal arms in $\mathbb{R}^3$, then we have a surjective map (generically 2^n-to-1)

$$\mathrm{FrHopf}: \mathbb{H}^n \longrightarrow \mathrm{FArm}(n)$$

$$(q_1, \dots, q_n) \longmapsto \left(\begin{bmatrix} \bar{q}_1 i q_1 & \bar{q}_1 j q_1 & \bar{q}_1 k q_1 \end{bmatrix}, \dots, \begin{bmatrix} \bar{q}_n i q_n & \bar{q}_n j q_n & \bar{q}_n k q_n \end{bmatrix} \right)$$

(The lingo is that $\mathbb{H}^n$ is then the moduli space of imploded, spin-framed polygonal arms; "imploded" b/c we allow edges of length 0, &

"spin-framed" b/c the frames are really in $Spin(3)$ rather than $SO(3)$)