Math 670: Day 34

We saw last time that the moduli space of planar n-gons up to translation & scaling is the Stiefel manifold $V_2\mathbb{R}^n$.

The next question is: How do we mod out by rotation?

Prop (Hausmann–Knutson, 1997): Rotating the polygon P given by $(\vec{a},\vec{b}) \in V_2\mathbb{R}^n$ by an angle φ has square root description

given by $\left(\cos(\varphi/2)\,\vec{a} - \sin(\varphi/2)\,\vec{b},\ \sin(\varphi/2)\,\vec{a} + \cos(\varphi/2)\,\vec{b}\right) \in V_2\mathbb{R}^n$.

Cor: The moduli space of planar polygons of perimeter 2 (or up to scaling) up to translation & rotation is the Grassmannian

$G_2\mathbb{R}^n$.

Proof of Prop: If $\vec{e}_k = w_k^2 = (r_k e^{i\theta_k})^2 = r_k^2 e^{i2\theta_k}$, then rotating by φ gives a new k^{th} edge

$$r_k^2 e^{i(2\theta_k+\varphi)} = r_k^2 e^{i2(\theta_k+\varphi/2)} = \left(r_k e^{i(\theta_k+\varphi/2)}\right)^2 = u_k^2,$$

and then the square root is

$$u_k = r_k e^{i(\theta_k+\varphi/2)} = r_k\cos(\theta_k+\varphi/2) + i\,r_k\sin(\theta_k+\varphi/2) = r_k\cos\theta_k\cos(\varphi/2) - r_k\sin\theta_k\sin(\varphi/2) + i\left(r_k\sin\theta_k\cos(\varphi/2) + r_k\cos\theta_k\sin(\varphi/2)\right)$$

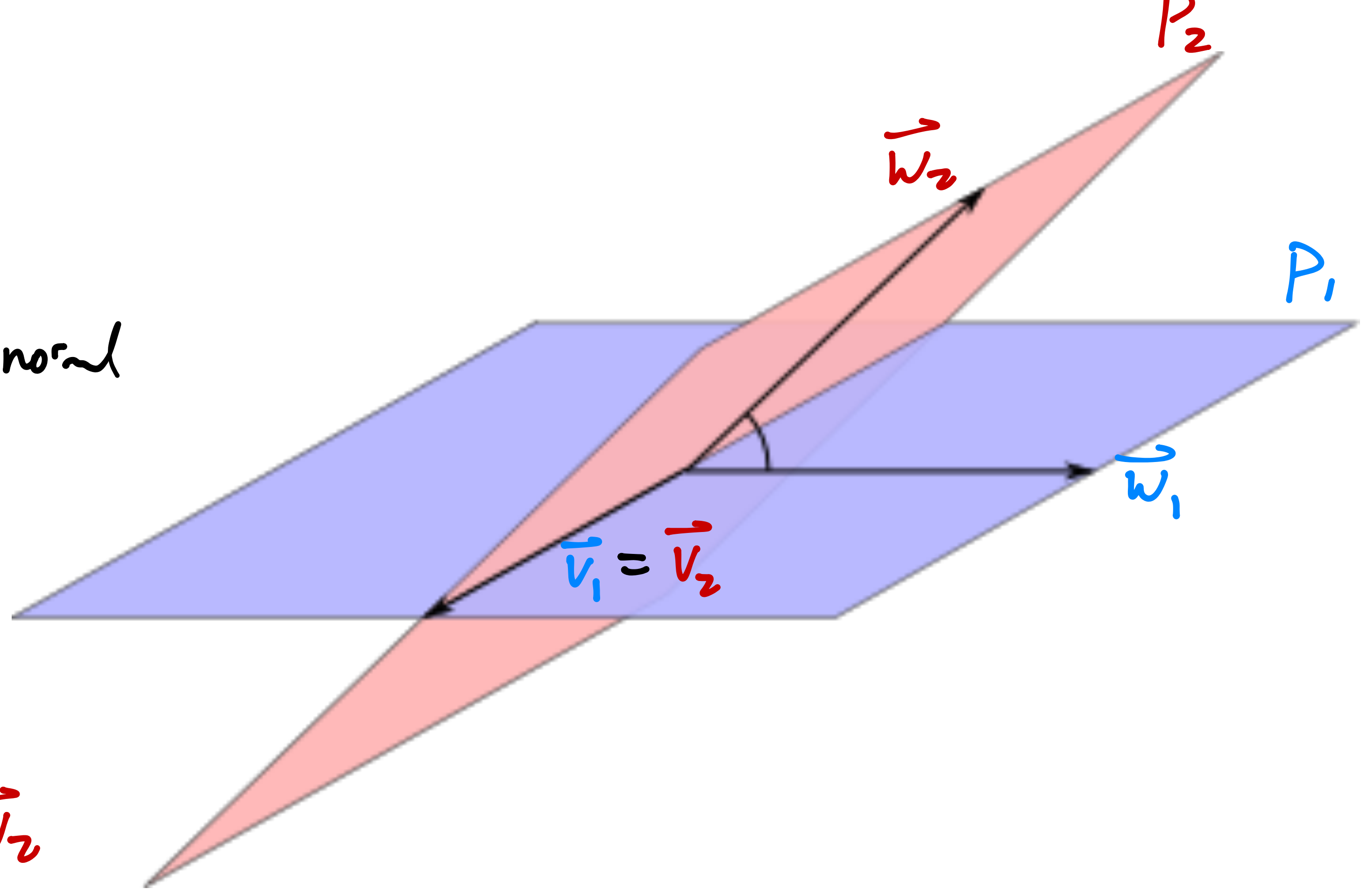
$$= a_k\cos\varphi/2 - b_k\sin\varphi/2 + i\left(b_k\cos\varphi/2 + a_k\sin\varphi/2\right)$$

$$= a_k\cos\varphi/2 - b_k\sin\varphi/2 + i\left(a_k\sin\varphi/2 + b_k\cos\varphi/2\right),$$

so the new square-root description is as described. $\blacksquare$

So now the point is that to compare shapes, we should look at distances in $G_2\mathbb{R}^n$.

Okay, well how far apart are two points in $G_2\mathbb{R}^n$?



Thm (Jordan): Any two planes P_1, P_2 in $\mathbb{R}^n$ have a pair of orthonormal

bases $\vec{v}_1, \vec{w}_1$ & $\vec{v}_2, \vec{w}_2$ s.t.

① $\vec{v}_2$ minimizes the angle b/w $\vec{v}_1$ & any unit vector in P_2, &
$\vec{w}_2$ minimizes the angle b/w $\vec{w}_1 \perp \vec{v}_1$ & any vector in P_2 perp. to $\vec{v}_2$

② vice versa.

The angles θ_1 b/w $\vec{v}_1$ & $\vec{v}_2$ and θ_2 b/w $\vec{w}_1$ & $\vec{w}_2$ are called the Jordan angles b/w the planes & the rotation carrying $\vec{v}_1 \to \vec{v}_2$ & $\vec{w}_1 \to \vec{w}_2$ is called the direct rotation of P_1 to P_2 & is the shortest path in $G_2\mathbb{R}^n$ b/w them.

In particular, the distance b/w P_1 & P_2 is $\sqrt{\theta_1^2+\theta_2^2}$.

Moreover, this is all easy to compute: If $\Pi_1 : P_1 \to P_1$ is the composition of the orthogonal projection $P_1 \to P_2$ & the orthogonal projection $P_2 \to P_1$, then the ON basis $\vec{v}_1, \vec{w}_1$ for P_1 are the eigenvectors for Π_1, & the eigenvalues of Π_1 are $\cos^2\theta_1$ & $\cos^2\theta_2$.

Likewise if we apply the transposition $(1\ 2)$ to the previous sentence.

Notice: This also gives us a preferred orientation for comparing two polygons. If the polygons correspond to $(\vec{a}_1, \vec{b}_1), (\vec{a}_2, \vec{b}_2) \in V_2\mathbb{R}^n$ & we let $P_i = \text{span}\{\vec{a}_i, \vec{b}_i\}$, then the rotations sending $(\vec{a}_1, \vec{b}_1) \mapsto (\vec{v}_1, \vec{w}_1)$ and $(\vec{a}_2, \vec{b}_2) \mapsto (\vec{v}_2, \vec{w}_2)$ correspond to the rigid rotations in the plane matching the polygons as closely (w.r.t. the metric on $V_2\mathbb{R}^n$) as possible.

In other words, this is a way of solving the curve registration problem

(Mathematica demo)