Math 670: Day 33

To parametrize closed polygons in $\mathbb{R}^2$, the key idea is to take square roots of the edges.

In other words, for any polygon in $\mathbb{R}^2$, we can think of the polygon up to translation as a list of edges $\vec{e}_1, \ldots, \vec{e}_n$. Now, treat each $\vec{e}_i$ as a complex number & choose $w_i \in \mathbb{C}$ s.t. $w_i^2 = \vec{e}_i$ (of course, there are 2^n possible choices).

So given a polygon (open or closed), we get $(w_1, \ldots, w_n) \in \mathbb{C}^n$, which of course we could also write as

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 i \\ \vdots \\ a_n + b_n i \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + i \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \vec{a} + \vec{b} i, \text{ for } \vec{a}, \vec{b} \in \mathbb{R}^n.$$

Now the magical fact:

Prop (Hausmann–Knutson, 1997): The polygon is closed $\Leftrightarrow$ $\vec{a}$ & $\vec{b}$ are orthogonal & the same length.

Pf: Since $\vec{e}_k = w_k^2 = (a_k + i b_k)(a_k + i b_k) = a_k^2 - b_k^2 + 2 a_k b_k i$, we see that the polygon is closed $\Leftrightarrow$

$$0 = \sum \vec{e}_k = \sum w_k^2 = \sum (a_k^2 - b_k^2 + 2 a_k b_k i) = \|\vec{a}\|^2 - \|\vec{b}\|^2 + 2\langle \vec{a}, \vec{b} \rangle i, \text{ which means}$$

$$\|\vec{a}\|^2 - \|\vec{b}\|^2 = 0 \;\&\; \langle \vec{a}, \vec{b} \rangle = 0.$$ ☒

Also:

Prop (Hausmann–Knutson 1997): The perimeter of the polygon is $\|\vec{a}\|^2 + \|\vec{b}\|^2$.

Pf: The perimeter of the polygon is

$$\sum \|\vec{e}_k\| = \sum |w_k|^2 = \sum |w_k|^2 = \sum (|a_k|^2 + |b_k|^2) = \|\vec{a}\|^2 + \|\vec{b}\|^2.$$ ☒

Therefore, the space of planar n-edge closed polygons of total perimeter 2 is (almost) 2^n-fold covered by the Stiefel manifold $V_2 \mathbb{R}^n$.

So the right way to compare planar polygons up to translation & scaling is by computing distances in the Stiefel manifold

(Computer examples, plus digression on distances in the Stiefel manifold)