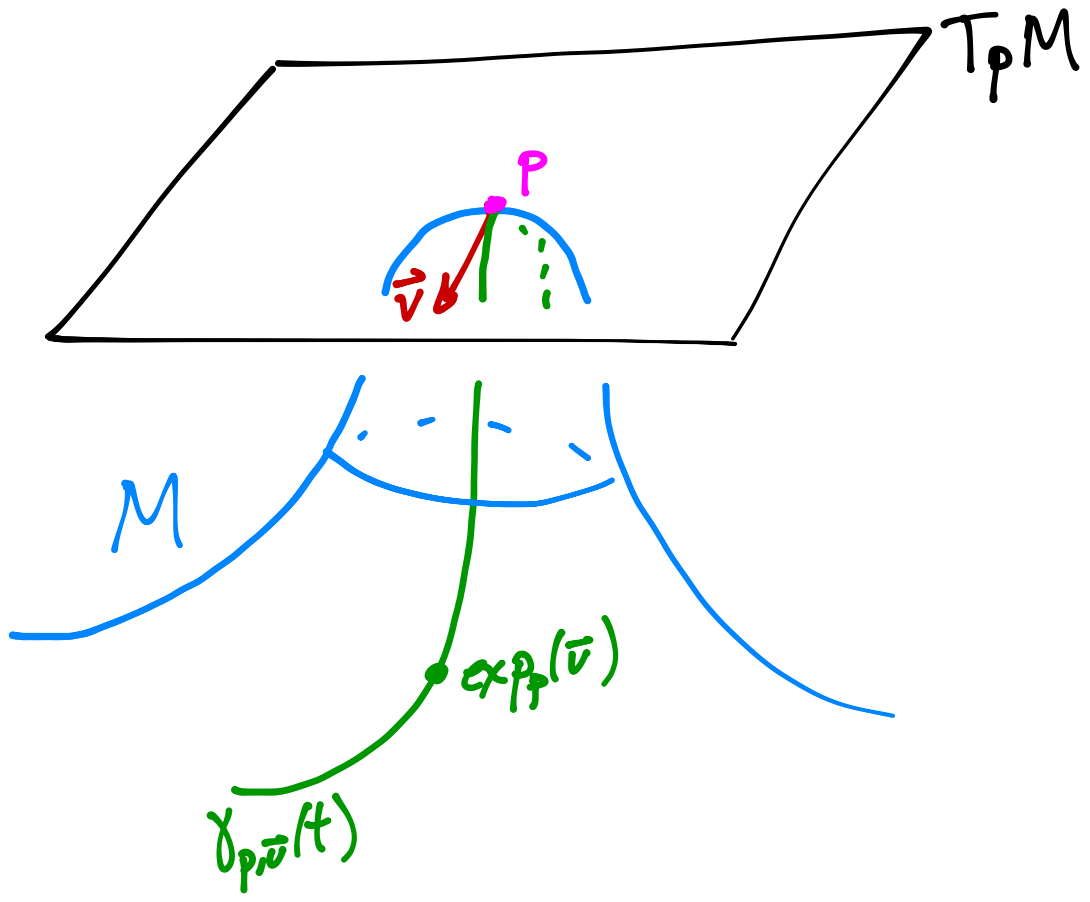
Math 670: Day 31

There is the following standard construction in differential geometry:

Given a vector $\vec{v} \in T_pM$, define $\exp_p(\vec{v})$ to be the endpoint of the geodesic of length $|\vec{v}|$ passing thru $\vec{v}$ & going in the $\vec{v}/|\vec{v}|$ direction.

In other words, if $\gamma_{p,\vec{v}}(t)$ is a geodesic in M s.t. $\gamma_{p,\vec{v}}(0) = p$ & $\gamma'_{p,\vec{v}}(0) = \vec{v}$, then $\exp_p(\vec{v}) = \gamma_{p,\vec{v}}(1)$.



In HW you saw that the matrix exponential map from $\mathfrak{so}(n) \to SO(n)$ is well-defined & surjective. In fact, this is a general feature of matrix gps &, moreover, the matrix exponential is the above exponential map

Thm: If G is a matrix Lie group, then the matrix exponential map $\exp: \mathfrak{g} \to G$ coincides with the Riemannian exponential map at the identity.

Useful facts about matrix exponentials:

Lemma: If A is a square matrix & U is invertible, then $e^{UAU^{-1}} = Ue^AU^{-1}$.

Pf: $(UAU^{-1})^n = UA^nU^{-1}$, so $e^{UAU^{-1}} = \sum \frac{(UAU^{-1})^n}{n!} = U \sum \frac{A^n}{n!} U^{-1} = Ue^AU^{-1}$. ☒

Recall the

Schur Decomposition Thm: Any complex matrix A can be written as $A = UTU^*$ where T is upper-triangular & U is unitary.

Prop: Given any complex matrix A, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then $e^{\lambda_1}, \dots, e^{\lambda_n}$ are the eigenvalues of e^A. Further, if $\vec{v}$ is an eigenvector of A w/ eigenvalue λ_i, then $\vec{v}$ is also an eigenvector for e^A w/ eigenvalue e^{λ_i}.

Pf: Since $A = UTU^*$, Lemma $\Rightarrow$ $e^A = e^{UTU^*} = e^{UTU^{-1}} = Ue^TU^{-1}$, so e^A & e^T have same eigenvalues.

Now, the diagonal entries of T^k are just T_{ii}^k, so the diagonal entries of e^T are just $e^{\lambda_1}, \dots, e^{\lambda_n}$.

As for eigenvectors, if $A\vec{v} = \lambda \vec{v}$, then $A^k \vec{v} = \lambda^k \vec{v}$, so $e^A \vec{v} = e^{\lambda}\vec{v}$. $\boxtimes$

Cor: $\det(e^A) = \det(e^T) = e^{\lambda_1}\cdots e^{\lambda_n} = e^{\lambda_1+\cdots+\lambda_n} = e^{\operatorname{tr} A}$. In particular, e^A is always invertible

(In particular, $\exp : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to GL(n,\mathbb{R})$ is well-defined)

Lemma: $(e^A)^{-1} = e^{-A}$

Pf: In HW you showed that if A & B commute, then $e^{A+B} = e^A e^B$. Since A & $-A$ commute, $e^A e^{-A} = e^{A-A} = e^0 = I$. $\boxtimes$

Ex: Consider $\exp : \mathfrak{so}(3) \to SO(3)$. Let $A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ & define $\theta = \sqrt{a^2+b^2+c^2}$ & $B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$.

Claim: $e^A = \cos\theta\, I + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B = I + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2$ if $\theta \neq 0$.

In particular, e^A is the rotation around the axis (a,b,c) by an angle θ, and hence we see that lines in $\mathfrak{so}(3)$ map to rotations around a fixed axis in $SO(3)$. Hence, by the Thm, these are exactly the geodesics thru the identity in $SO(3)$.

Of course, the same is true in general: if $\Delta \in \mathfrak{so}(n) = \mathfrak{o}(n)$ (i.e., Δ is skew-symmetric), then the corresponding geodesic in $SO(n)$ or $O(n)$ is $\exp(t\Delta) = e^{t\Delta}$ is the rotation around a fixed axis.

In turn, since left-translation takes the identity to any other point $Q \in O(n)$ in a geometry-preserving way, geodesics thru $Q \in O(n)$ are of the form $Q e^{t\Delta}$

Now, recall from before I left that, if
$$\begin{tikzcd} & O(n) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V_k\mathbb{R}^n & & G_k\mathbb{R}^n \end{tikzcd}$$
are the obvious projections & $Q \in O(n)$, then the tangent vectors to $\pi_1(Q) \in V_k\mathbb{R}^n$ look like $Q\begin{bmatrix} 0 & -B^T \\ B & A \end{bmatrix}$ where $A \in O(k)$ & B is $k \times (n-k)$.

Likewise, tangent vectors to $\pi_2(Q) \in G_k\mathbb{R}^n$ look like $Q\begin{bmatrix} 0 & -B^T \\ B & 0 \end{bmatrix}$ for B a $k\times(n-k)$ matrix.

So now if we want geodesics in $V_k\mathbb{R}^n$ or $G_k\mathbb{R}^n$, we just restrict the appropriate choices of Δ:

Geodesics thru $\pi_1(Q) \in V_k\mathbb{R}^n$ are of the form $\pi_1(Q)\, e^{t\begin{bmatrix} 0 & -B^T \\ B & A \end{bmatrix}}$ & geodesics thru $\pi_2(Q) \in G_k\mathbb{R}^n$ are of the form $\pi_2(Q)\, e^{t\begin{bmatrix} 0 & -B^T \\ B & 0 \end{bmatrix}}$.