Math 670: Day 30

I want to talk about geodesics in $G_k\mathbb{R}^n$ & $V_k\mathbb{R}^n$, and they lead into connections b/w these spaces & moduli spaces of closed random walks, but to do so I first want to back up a bit & talk about some useful linear algebra theorems.

First, a theorem which shows we can think of $O(n)$ (or $U(n)$) as the rigid, compact core of the much floppier $GL(n,\mathbb{R})$ (or $GL(n,\mathbb{C})$):

Thm: $GL(n,\mathbb{R})$ deformation retracts to $O(n)$.

Proof: Gram-Schmidt is clearly continuous & preserves orthogonal matrices. Since it can be done as a deformation, we're done. ☐

In fact, there's rather a better proof, which is directly analogous to the fact that $\mathbb{C}^*$ deformation retracts to the unit circle.

Recall that complex numbers $a+bi$ can be written in "polar form" as $a+bi = re^{i\theta}$. Analogously:

Thm: For any real $n\times n$ invertible matrix A, $\exists!$ orthogonal matrix R & positive-definite symmetric matrix S s.t.

$$A = RS.$$

(Likewise for complex A, where R is unitary).

For a proof, see Halmos §83... basically, let $S=\sqrt{A^*A}$. If A invertible then can let $R=(SA^{-1})^{-1}$. Since

$(SA^{-1})^*SA^{-1} = (A^{-1})^*S^*SA^{-1} = (A^*)^{-1}SSA = (A^*)^{-1}A^*AA^{-1} = I_n$, R^{-1} (& hence R) is orthogonal. If A not invertible, have to work harder.

Thm (Cholesky Decomposition): Any symmetric positive-definite matrix S can be written uniquely as $S=LL^*$ where L is a lower-triangular matrix with strictly positive diagonal entries.

Cor: $GL(n,\mathbb{R}) \cong O(n)\times\mathbb{R}^{\frac{n(n+1)}{2}}$

$GL(n,\mathbb{C}) \cong U(n)\times\mathbb{R}^{n^2}$ (b/c lower-triangular complex matrices w/ positive diagonal entries have dimension $2\frac{n(n-1)}{2}+n = n^2$)

Now, the connection to Stiefel & Grassmann manifolds:

Clearly, $V_n(\mathbb{R}^n)=O(n)$ & $V_n(\mathbb{C}^n)=U(n)$. For other k, we have:

Lemma: $V_k\mathbb{R}^n \cong \{A \in n\times k \text{ matrices} : A^TA = I_k\}$

Pf: A^TA is the Gram matrix of A; i.e., the matrix of dot products of the columns of A. Clearly it is $I_k \Leftrightarrow$ the columns form an orthonormal k-frame. ☐

As a consequence:

Thm: For every $n\times k$ matrix A of full rank, there are unique matrices $R\in V_k\mathbb{R}^n$ & $S\in$ Symm. pos. def. $k\times k$ matrices s.t. $A=RS$.

Combined w/ the Cholesky decomposition as before, we get

Cor: Up to homeomorphism,

$$\text{rank } k,\ n\times k \text{ matrices} \cong \begin{cases} V_k\mathbb{R}^n \times \mathbb{R}^{\frac{k(k+1)}{2}} \\ V_k\mathbb{C}^n \times \mathbb{R}^{k^2} \end{cases}$$

depending on whether they are real or complex.

In other words, topologically the key feature of the Stiefel manifolds is the linear independence of the frame, & not the orthonormality.

The foregoing also gives important numerical insight into the Stiefel manifolds.

Def: If A is an $n\times n$ matrix, let the Frobenius norm of A be defined by $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\operatorname{trace}(A^*A)}$

Thm: Given an $n\times k$ matrix A of rank k, the nearest matrix (w.r.t. the Frobenius norm) to A in $V_k\mathbb{R}^n$ is the matrix R given by the polar decomposition (& similarly for complex matrices & $V_k\mathbb{C}^n$).

In other words, when doing numerical calculations on the Stiefel manifold, we have a natural numerical method for correcting any errors which accumulate during a calculation to make the frame non-orthonormal: compute the polar decomposition.