Math 670: Day 3

Recall that last time we defined a tangent vector as, essentially, a linear derivation on the space of differentiable functions, which is a nice coordinate-free way of defining things, but not necessarily so useful for computation.

After all, one does not typically compute directly from the definition

Instead, you usually use local coordinates for computations, so let's see what this all means in local coords:

If (U, φ) is a coord. chart containing $p \in M$ s.t. $\varphi(\vec{0}) = p$, then

$$f \circ \varphi(q) = f(x_1, \ldots, x_n) \quad \text{for } q = (x_1, \ldots, x_n) \in U \subseteq \mathbb{R}^n$$

and

$$\varphi^{-1} \circ \alpha(t) = (x_1(t), \ldots, x_n(t))$$

so

$$\alpha'(0) f = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt} f(x_1(t), \ldots, x_n(t))\Big|_{t=0}$$

$$= \sum_{i=1}^{n} x_i'(0) \frac{\partial f}{\partial x_i} = \left(\sum_i x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_{\vec{0}} \right) f .$$

In other words, we can write $\alpha'(0)$ in local coords. as

$$\alpha'(0) = \sum_i x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_{\vec{0}}$$

The tangent vectors $\left(\frac{\partial}{\partial x_i}\right)_{\vec{0}}$ give a basis of T_pM associated to the chart (U, φ).

Def: The tangent bundle of a manifold M^n, denoted TM, is the union of the tangent spaces at each point:

$$TM = \bigcup_{p \in M} T_pM .$$

Likewise, if T_p^*M is the dual space to T_pM (i.e., the space of linear functionals on T_pM, called the cotangent space), then

$$T^*M = \bigcup_{p \in M} T_p^*M$$

is the cotangent bundle of M.

There is a projection $\pi : TM \to M$ given by $\pi(p, v) = p$.

Prop: TM is a $2n$-dimensional mfd.

Proof: Let $\{(U_\alpha, \varphi_\alpha)\}$ be a maximal diff'ble structure (collection of coord. charts) on M. Let $(x_1^\alpha, \dots, x_n^\alpha)$ be the coords of U_α & let $\{\frac{\partial}{\partial x_1^\alpha}, \dots, \frac{\partial}{\partial x_n^\alpha}\}$ be the associated bases for the tangent spaces of $\varphi_\alpha(U_\alpha)$. Define

$$\psi_\alpha : U_\alpha \times \mathbb{R}^n \to TM$$

$$(x_1^\alpha, \dots, x_n^\alpha, t_1, \dots, t_n) \mapsto \left(\varphi_\alpha(x_1^\alpha, \dots, x_n^\alpha), \sum t_i \frac{\partial}{\partial x_i^\alpha}\right)$$

Then $\{(U_\alpha \times \mathbb{R}^n, \psi_\alpha)\}$ gives a differentiable structure on TM. ∎

Ex:
- $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$
- $TS^1 \cong$ cylinder
- TS^2 is a non-trivial bundle over S^2. The **unit tangent bundle** UTS^2 (consisting of unit tangent vectors) is homeomorphic to $SO(3) \cong \mathbb{R}P^3$, which is **not** the same as $S^2 \times S^1$.

Def: Suppose M^m & N^n are differentiable mfds & $f: M \to N$ is smooth. For each $p \in M$ define the **differential** of f at p, denoted $df_p : T_pM \to T_{f(p)}N$ as follows:

for each $v \in T_pM$, choose a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to M$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$. Let $\beta = f \circ \alpha$. Then

$$df_p(v) := \beta'(0).$$

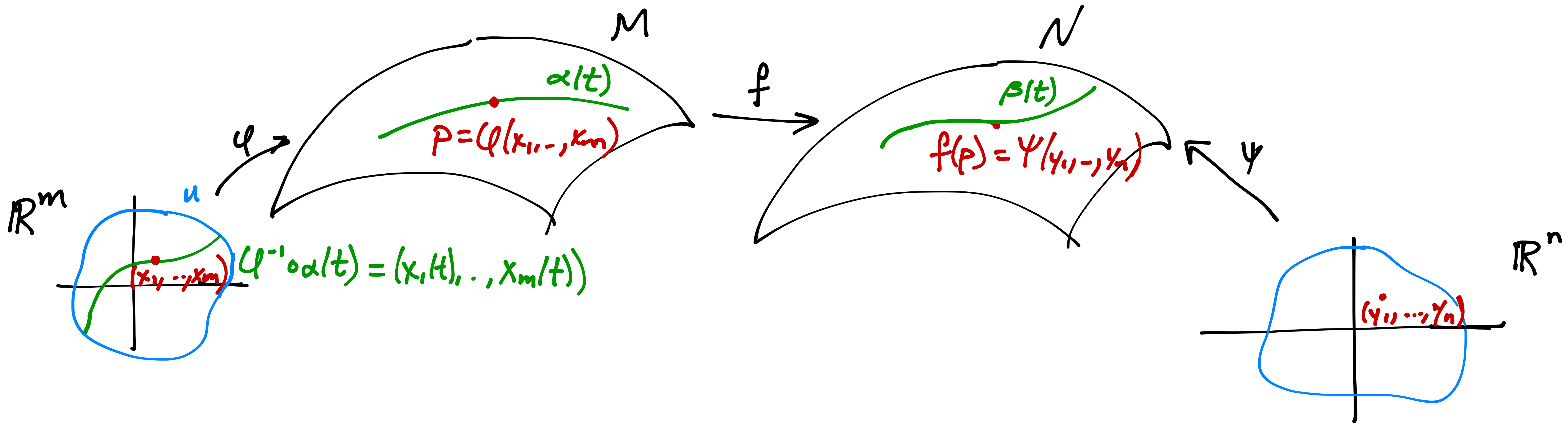
Lemma: This is a well-defined linear map.

Proof: Notice that

$$df_p(v)(g) = \frac{d(g \circ \beta)}{dt}\Big|_{t=0} = \frac{d(g \circ f \circ \alpha)}{dt}\Big|_{t=0} = \frac{d((g \circ f) \circ \alpha)}{dt}\Big|_{t=0} = \alpha'(0)(g \circ f) = v(g \circ f),$$

which is clearly linear & depends only on $\alpha'(0) = v$ & not anything else about α. ∎

In coords:



$$(y_1, \dots, y_n) = \psi^{-1} \circ f \circ \varphi(x_1, \dots, x_n) = (y_1(x_1, \dots, x_m), \dots, y_n(x_1, \dots, x_m))$$

Then

$$d(\psi^{-1})_{f(p)} \beta'(0) = \left(\sum \frac{\partial y_1}{\partial x_i} x_i'(0), \dots, \sum \frac{\partial y_n}{\partial x_i} x_i'(0) \right) = \underbrace{\left(\frac{\partial y_i}{\partial x_j} \right)}_{\substack{n \times m \text{ matrix} \\ \text{"}df_p\text{"}}} \underbrace{\left(x_j'(0) \right)}_{\substack{m\text{-element column vector} \\ \text{"}v\text{"}}}$$