Math 670: Day 29

Stiefel & Grassmann mflds as Homogeneous Spaces

- Let $V_k(\mathbb{R}^n)$ be the Stiefel mfld of k-frames in $\mathbb{R}^n$: $V_k(\mathbb{R}^n) := \{(v_1,\dots,v_k) : v_i \in \mathbb{R}^n,\ \langle v_i, v_j\rangle = \delta_{ij}\ \forall i,j\}$.

  Then $O(n)$ acts transitively on $V_k(\mathbb{R}^n)$ (so long as $k<n$).

  Consider the frame $(e_{n-k+1},\dots,e_n) \in V_k(\mathbb{R}^n)$. Then the isotropy subgp is

  $$\left\{ \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_k \end{array}\right) : M \in O(n-k) \right\} \cong O(n-k).$$

  So we see that $V_k(\mathbb{R}^n) \cong O(n)/O(n-k)$

- Let $G_k(\mathbb{R}^n)$ be the Grassmann mfld of k-planes in $\mathbb{R}^n$. Again, $O(n)$ acts transitively, & the isotropy subgp of $\langle e_{n-k+1},\dots,e_n\rangle$ is $\cong O(k)\times O(n-k)$, so $G_k(\mathbb{R}^n) \cong \dfrac{O(n)}{O(k)\times O(n-k)}$

  Notice that, in particular, $G_k(\mathbb{R}^n) \cong V_k(\mathbb{R}^n)/O(k) \cong \dfrac{O(n)}{O(k)\times O(n-k)} \cong V_{n-k}(\mathbb{R}^n)/O(n-k) \cong G_{n-k}(\mathbb{R}^n)$

Now, $\dim V_k(\mathbb{R}^n) = \dim O(n) - \dim O(n-k) = \dfrac{n(n-1)}{2} - \dfrac{(n-k)(n-k-1)}{2} = \dfrac{2nk-k^2-k}{2} = nk - \dfrac{k(k+1)}{2}$

$\dim G_k(\mathbb{R}^n) = \dim V_k(\mathbb{R}^n) - \dim O(k) = nk - \dfrac{k(k+1)}{2} - \dfrac{k(k-1)}{2} = nk - k^2 = k(n-k)$.

Notice that $\dim V_n(\mathbb{R}^n) = n^2 - \dfrac{n(n+1)}{2} = \dfrac{n(n-1)}{2}$, which makes sense since $V_n(\mathbb{R}^n) = O(n)$.

- Same construction works for complex Stiefel & Grassmann mflds:

  $$V_k(\mathbb{C}^n) \cong U(n)/U(n-k) \quad \& \quad G_k(\mathbb{C}^n) \cong \frac{U(n)}{U(k)\times U(n-k)}$$

  so $\dim V_k(\mathbb{C}^n) = \dim U(n) - \dim U(n-k) = n^2 - (n-k)^2 = 2kn - k^2 = k(2n-k)$

  $\dim G_k(\mathbb{C}^n) = \dim V_k(\mathbb{C}^n) - \dim U(k) = 2kn - k^2 - k^2 = 2k(n-k)$.

Note: We get principal bundles

$$\begin{tikzcd} O(k) \arrow[r, hook] & V_k(\mathbb{R}^n) \arrow[d] \\ & G_k(\mathbb{R}^n) \end{tikzcd} \quad \& \quad \begin{tikzcd} U(k) \arrow[r, hook] & V_k(\mathbb{C}^n) \arrow[d] \\ & G_k(\mathbb{C}^n) \end{tikzcd}$$

Called the Stiefel bundle (the frame bundle associated to the tautological bundle of the Grassmannian)

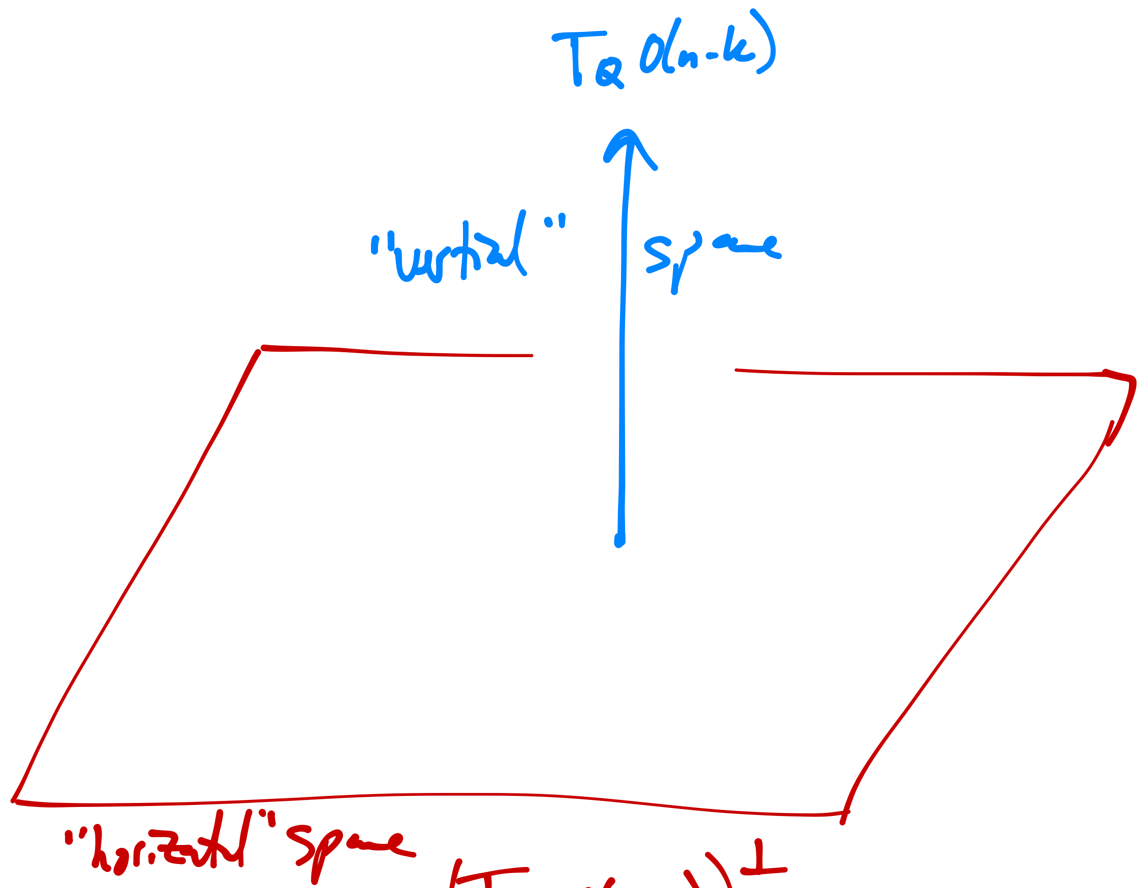
Let $k=1$ & focus on $\begin{array}{c} U(1) \hookrightarrow V_1(\mathbb{C}^n) \\ \downarrow \\ G_1(\mathbb{C}^n) \end{array}$ Now, $U(1)=S^1$, $G_1(\mathbb{C}^n)=\mathbb{C}P^{n-1}$, & $V_1(\mathbb{C}^n)=S^{2n-1}$, so this is just the generalized Hopf fibration $\begin{array}{c} S^1 \hookrightarrow S^{2n-1} \\ \downarrow \\ \mathbb{C}P^{n-1} \end{array}$ in disguise.

Likewise, the Hopf fibrations $\begin{array}{c} S^3 \hookrightarrow S^{4n-1} \\ \downarrow \\ \mathbb{H}P^{n-1} \end{array}$ generalize to Stiefel bundles $\begin{array}{c} Sp(k) \hookrightarrow V_k(\mathbb{H}^n) \\ \downarrow \\ G_k(\mathbb{H}^n) \end{array}$.

Now, consider the quotient map $\begin{array}{c} O(n-k) \hookrightarrow O(n) \\ \downarrow \pi \\ V_k(\mathbb{R}^n) \end{array}$.

Want to use this to think about the tangent space to $V_k(\mathbb{R}^n)$

$T_Q O(n)$ is the ambient space



$(T_Q O(n-k))^\perp$ = copy of tangent sp. to $V_k(\mathbb{R}^n)$ at $\pi(Q)$

The fiber $[Q]$ is the subset of matrices in $O(n)$ of the form $[Q]=\left\{Q\begin{pmatrix} B & 0 \\ 0 & I\end{pmatrix} : B\in O(n-k)\right\}$, which has tangent space

$$\mathrm{Vert}_Q = \left\{Q\begin{pmatrix} C & 0 \\ 0 & 0\end{pmatrix} : C \text{ is } (n-k)\times(n-k) \text{ skew-symmetric}\right\}$$

So then $T_{\pi(Q)} V_k(\mathbb{R}^n)$ is the orthogonal complement $\mathrm{Horiz}_Q = \left\{Q\begin{pmatrix} 0 & -B^* \\ B & A\end{pmatrix} : A \text{ is } k\times k \text{ skew-symm}, B \text{ is } k\times(n-k) \text{ arbitrary}\right\}$

So again we see $\dim V_k(\mathbb{R}^n) = k(n-k) + \frac{k(k-1)}{2} = nk - \frac{k(k+1)}{2}$.

Even better, this immediately tells us how to put a Riemannian metric on $V_k(\mathbb{R}^n)$: just restrict the Frobenius metric on $O(n)$ to the horizontal space:

If $\Delta_i = Q\begin{pmatrix} 0 & -B_i^* \\ B_i & A_i\end{pmatrix} \in T_Q V_k(\mathbb{R}^n)$, then

$$\langle \Delta_1, \Delta_2\rangle = \frac{1}{2}\operatorname{tr}\Delta_1^*\Delta_2 = \frac{1}{2}\operatorname{tr}\left(\left(Q\begin{pmatrix} 0 & -B_1^* \\ B_1 & A_1\end{pmatrix}\right)^*\left(Q\begin{pmatrix} 0 & -B_2^* \\ B_2 & A_2\end{pmatrix}\right)\right) = \frac{1}{2}\operatorname{tr}\begin{pmatrix} 0 & B_1^* \\ -B_1 & A_1^*\end{pmatrix}\begin{pmatrix} 0 & -B_2^* \\ B_2 & A_2\end{pmatrix}$$

$$= \frac{1}{2}\operatorname{tr} A_1^*A_2 + \frac{1}{2}\operatorname{tr} B_1^*B_2 + \frac{1}{2}\operatorname{tr} B_1B_2^*$$

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We can play the same game w/ $O(n)\to G_k(\mathbb{R}^n)$: $[Q]=\left\{Q\begin{pmatrix} B_{n-k} & 0 \\ 0 & B_k\end{pmatrix} : B_k\in O(k), B_{n-k}\in O(n-k)\right\}$ has tangent space

$\mathrm{Vert}_Q = \left\{\begin{pmatrix} A & 0 \\ 0 & C\end{pmatrix} : A\ (n-k)\times(n-k), C\ k\times k, \text{ both skew-symm.}\right\}$, so $T_{\pi(Q)} G_k(\mathbb{R}^n)$ is iso. to the orthog. complement

$\mathrm{Horiz}_Q = \left\{\begin{pmatrix} 0 & -B^* \\ B & 0\end{pmatrix} : B \text{ is } k\times(n-k)\right\}$. Clearly $\dim G_k(\mathbb{R}^n) = k(n-k)$ & the metric is given as follows:

If $\Delta_i = Q\begin{pmatrix} 0 & B_i^* \\ B_i & 0 \end{pmatrix} \in T_Q G_k(\mathbb{R}^n)$, then $\langle \Delta_1, \Delta_2 \rangle = tr\, B_1^* B_2$.