Math 670: Day 28

Ex: · Same reasoning as for S^2 gives that $S^n \cong SO(n+1)/SO(n)$ (or $O(n+1)/O(n)$).

· Consider $U(n)$ acting on the unit sphere S^{2n-1} in $\mathbb{C}^n$. Then the isotropy subgp of the north pole is the subgp

$$\left\{\left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right) : M \in U(n-1)\right\} \cong U(n-1), \text{ so } S^{2n-1} \cong U(n)/U(n-1)$$

Special Case: $S^1 \cong U(1)/U(0) = U(1)/\{1\} \cong U(1)$.

· Similarly, $S^{2n-1} \cong SU(n)/SU(n-1)$.

Special Case: $S^3 \cong SU(2)/SU(1) = SU(2)/\{1\} \cong SU(2)$.

· $\mathbb{CP}^n = \{(x_0,\dots,x_n) \in \mathbb{C}^{n+1}\setminus\{\vec{0}\}\}/\sim$ where $(x_0,\dots,x_n) \sim \lambda(x_0,\dots,x_n)$ $\forall\, \lambda \in \mathbb{C}$.

$= \{(x_0,\dots,x_n) \in S^{2n+1}\}/\sim$ where $(x_0,\dots,x_n) \sim \lambda(x_0,\dots,x_n)$ $\forall\, \lambda \in S^1$.

$SU(n+1)$ acts on S^{2n+1} & preserves the equivalence classes, so it acts (transitively) on $\mathbb{CP}^n$.

Isotropy subgp for, say $(0:\dots:0:1) \in \mathbb{CP}^n$ is just

$$\left\{\left(\begin{array}{c|c} M & 0 \\ \hline 0 & \frac{1}{\det(M)} \end{array}\right) : M \in U(n-1)\right\} \cong U(n), \text{ so } \mathbb{CP}^n \cong SU(n+1)/U(n).$$

Notice that, since $SU(n) \cong U(n)/U(1)$, we have that $\mathbb{CP}^n \cong SU(n+1)/U(n) \cong \left(SU(n+1)/SU(n)\right)/U(1) \cong S^{2n+1}/U(1)$,

giving a fibration
$$\begin{tikzcd} U(1) \arrow[r, hook] & S^{2n+1} \arrow[d] \\ & \mathbb{CP}^n \end{tikzcd}$$

When $n=1$, this gives
$$\begin{tikzcd} U(1) \arrow[r, hook] & S^3 \arrow[d] \\ & \mathbb{CP}^1 \cong S^2 \end{tikzcd}$$
The vertical map $S^3 \to S^2$ is better known as the Hopf fibration.

For $n>1$, the fibrations $S^1 \hookrightarrow S^{2n+1} \to \mathbb{CP}^n$ are generalized Hopf fibrations.

· Let $Sp(n)$ be the symplectic group, i.e., the subgp of $GL(n,\mathbb{H})$ that preserves the quaternionic inner product on $\mathbb{H}^n$; equivalently, the subgp of elements $M \in SU(2n)$ s.t. $M^*\Omega M = \Omega$, where $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Then $Sp(n+1)$ acts transitively on $S^{4n+3} \subseteq \mathbb{H}^{n+1}$ & preserves quaternionic lines, so acts on $\mathbb{HP}^n$.

The isotropy subgroup of a point in $\mathbb{HP}^n$ is iso. to $Sp(n)$, so

$$\mathbb{HP}^n \cong Sp(n+1)/(Sp(1)\times Sp(n))$$

Now, $S^{4n+3} \cong Sp(n+1)/Sp(n)$, so we see that $\mathbb{HP}^n \cong S^{4n+3}/Sp(1)$. Since $Sp(1) = S^3$, this gives another generalized Hopf fibration $S^3 \hookrightarrow S^{4n+3} \to \mathbb{HP}^n$. Since $\mathbb{HP}^1 \cong S^4$, we get the special case $S^3 \hookrightarrow S^7 \to S^4$.

There turns out to be 1 more Hopf fibration: $S^7 \hookrightarrow S^{15} \to S^8 \cong \mathbb{OP}^1$, but the non-associativity of the octonions prevents you from being able to define $\mathbb{OP}^n$ for $n > 2$ & it turns out that, even though $\mathbb{OP}^2$ exists, there's no fibration $S^7 \hookrightarrow S^{23} \to \mathbb{OP}^2$.