Math 670: Day 27

Ex: Consider $SO(3)$ acting on S^2 by rotations. Let $N=(0,0,1)$ be the north pole. Then the isotropy subgroup at N is

precisely the subgroup $H=\left\{\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} : \theta\in S^1\right\}\cong SO(2)$.

In fact, for any $\vec{x}\in S^2$, the isotropy subgroup at $\vec{x}$ is precisely $r_{\vec{x}} H r_{\vec{x}}^{-1}$ where $r_{\vec{x}}$ is the rotation matrix which rotates

N to $\vec{x}$ within the plane spanned by N & $\vec{x}$.

In fact, this is generally true: the isotropy subgroups of distinct points in M are conjugate in H.

Now, notice that $S^2\cong SO(3)/H$ (or, more colloquially, $S^2\cong SO(3)/SO(2)$)

To see this, define $\varphi: SO(3)/H \to S^2$ by

$$gH \mapsto g\cdot N$$

Well-defined: For $h\in H$, $\varphi(ghH) = gh\cdot N = g\cdot(h\cdot N) = g\cdot N$

Surjective: $SO(3)$ acts transitively on S^2, so $\forall\, \vec{x}\in S^2\ \exists\, g\in SO(3)$ s.t. $g\cdot N=\vec{x}$. But then $\varphi(g\cdot H)=g\cdot N=\vec{x}$.

Injective: Suppose $\varphi(gH)=\varphi(aH)$. Then $\varphi(a^{-1}gH)=a^{-1}g\cdot N = a^{-1}\cdot(g\cdot N) = a^{-1}\cdot(a\cdot N) = (a^{-1}a)\cdot N = I\cdot N = N$,

so $a^{-1}g\in H \Rightarrow aH = aa^{-1}gH = gH$.

Smooth: If $\pi: SO(3)\to SO(3)/H$ is the projection, then I claim φ is smooth $\Longleftrightarrow$ $\varphi\circ\pi$ is smooth

$$\begin{tikzcd} SO(3) \arrow[d, "\pi"'] \arrow[dr, dashed, "\varphi\circ\pi"] & \\ SO(3)/H \arrow[r, "\varphi"] & S^2 \end{tikzcd}$$

Clearly, if φ is smooth, then $\varphi\circ\pi$ is smooth since π is.

OTOH, if $\varphi\circ\pi$ is smooth, then, on a nbhd W of $gH\in SO(3)/H$ w/ local section $\sigma: W\to G$, we have that

$$(\varphi\circ\pi)\circ\sigma = \varphi\circ(\pi\circ\sigma) = \varphi\circ id_W = \varphi|_W \text{ is smooth.}$$

Since such local sections exist at all points in $SO(3)/H$, φ is smooth.

But now $\varphi\circ\pi(g) = g\cdot N$ is smooth since the Lie group action is smooth, & we see that

φ is smooth.

Smooth inverse: By the Inverse Function Thm, it suffices to show that $d\varphi$ is nonsingular or, equivalently, that

$$\ker d(\varphi\circ\pi)_g \subseteq T_gH \subseteq T_g SO(3)$$

of course $\varphi\circ\pi = \eta_g \circ (\varphi\circ\pi)\circ L_{g^{-1}}$, so it suffices to show this at the identity: $d(\varphi\circ\pi)_I \subseteq T_I H$.

But now, if $A = \begin{pmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{pmatrix} \in T_I SO(3)$, then $A = \alpha'(0)$ where $\alpha(t) = e^{tA}$, so

$$d(\varphi\circ\pi)_I A = d(\varphi\circ\pi)_I\, \alpha'(0) = \frac{d}{dt}\Big|_{t=0} \varphi\circ\pi\circ\alpha(t) = \frac{d}{dt}\Big|_{t=0} (\varphi\circ\pi)\, e^{tA} = \frac{d}{dt}\Big|_{t=0} e^{tA}\cdot N$$

$$= \frac{d}{dt}\Big|_{t=0} e^{t\begin{pmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \dots = \begin{pmatrix} z \\ y \\ 0 \end{pmatrix}.$$

So $A \in \ker(\varphi\circ\pi) \iff z = y = 0 \iff A = \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which is indeed in the tangent space to H.

Therefore, $d\varphi$ is nonsingular everywhere & hence has a smooth inverse.

Essentially the same proof gives the following theorem:

Thm: Suppose a Lie gp. G acts transitively on a mfd M & that, for $x \in M$, H is the isotropy subgp for x. Then M is diffeomorphic to G/H.