Math 670: Day 26

Notice that in our proof of the previous thm, the fact that the ideal $\mathcal{I}$ generated by

$$\mu_i := \pi_1^* \varphi^* \omega_i - \pi_2^* \omega_i$$

is a differential ideal didn't depend at all on the Lie gp. homomorphism $\varphi: G \to H$; it only depended on $\varphi^*: \mathfrak{h}^* \to \mathfrak{g}^*$

or, equivalently, on the Lie alg. homomorphism $d\varphi: \mathfrak{g} \to \mathfrak{h}$.

This is the key point in proving the following theorem:

Thm: Let G & H be Lie gps w/ G simply connected. If $\psi: \mathfrak{g} \to \mathfrak{h}$ is a Lie alg. homomorphism, then $\exists!$ Lie gp. hom.

$\varphi: G \to H$ s.t. $d\varphi = \psi: \mathfrak{g} \to \mathfrak{h}$.

Cor: If simply connected Lie gps G & H have isomorphic Lie algebras, then G & H are isomorphic.

The simply-connected hypothesis is essential:

Ex: $O(n)$ & $SO(n)$ have identical Lie algebras even though they're not isomorphic as Lie gps (nor even homeomorphic).

More subtly:

Ex: $SO(3)$ & $SU(2)$ have isomorphic Lie algebras even though they're not isomorphic as Lie gps (nor even homeomorphic).

To see this recall that $\mathfrak{so}(3) = \{3\times 3 \text{ skew-symmetric matrices}\} = \left\{ \begin{pmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$.

Now for $\mathfrak{su}(2)$: tangent vectors to the identity: let $\alpha: (-\varepsilon,\varepsilon) \to SU(2)$ s.t. $\alpha(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\alpha(t) \in SU(2)$ $\forall$ t, so

$$\alpha(t)\, \overline{\alpha(t)}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Differentiate: $\alpha'(t)\, \overline{\alpha(t)}^T + \alpha(t)\, \overline{\alpha'(t)}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Evaluate @ $t=0$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \alpha'(0)\,\overline{\alpha(0)}^T + \alpha(0)\,\overline{\alpha'(0)}^T = \alpha'(0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\overline{\alpha'(0)}^T = \alpha'(0) + \overline{\alpha'(0)}^T,$$

so the tangent vectors to the identity are skew-Hermitian. Of course, the space of 2×2 skew-Hermitian matrices is

four (real) dimensional, so we need another condition. Of course, that condition should come from differentiating

$$1 = \det \alpha(t) = \alpha_{11}(t)\alpha_{22}(t) - \alpha_{21}(t)\alpha_{12}(t)$$

$\overset{\frac{d}{dt}}{\rightsquigarrow}$ $0 = \alpha_{11}'(t)\alpha_{22}(t) + \alpha_{11}(t)\alpha_{22}'(t) - \alpha_{21}'(t)\alpha_{12}(t) - \alpha_{21}(t)\alpha_{12}'(t)$

At $t=0$: $\alpha_{11}(0)=1, \alpha_{12}(0)=0, \alpha_{21}(0)=0, \alpha_{22}(0)=1$, so get

$$0=\alpha_{11}'(0)+\alpha_{22}'(0).$$

So then we see that $\mathfrak{su}(2)$ is the set of skew-Hermitian, traceless 2×2 matrices

$$\mathfrak{su}(2)=\left\{\begin{pmatrix} ri & z\\ -\bar z & -ri\end{pmatrix}: r\in\mathbb{R}, z\in\mathbb{C}\right\}.$$

Now, the claim is that $\mathfrak{so}(3)\cong\mathfrak{su}(2)$. Define $\varphi:\mathfrak{so}(3)\longrightarrow\mathfrak{su}(2)$ by

$$\begin{pmatrix} 0 & x & z\\ -x & 0 & y\\ -z & -y & 0\end{pmatrix}\longmapsto \frac{1}{2}\begin{pmatrix} zi & x+iy\\ -x+iy & -zi\end{pmatrix}$$

Then, if $A=\begin{pmatrix} 0 & x_1 & z_1\\ -x_1 & 0 & y_1\\ -z_1 & -y_1 & 0\end{pmatrix}$, $B=\begin{pmatrix} 0 & x_2 & z_2\\ -x_2 & 0 & y_2\\ -z_2 & -y_2 & 0\end{pmatrix}$, we have

$$\varphi([A,B]) = \varphi\begin{pmatrix} 0 & y_1z_2-y_2z_1 & x_1y_2-x_2y_1\\ \cdot & 0 & x_2z_1-x_1z_2\\ \cdot & \cdot & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix} (x_1y_2-x_2y_1)i & (y_1z_2-y_2z_1)+(x_2z_1-x_1z_2)i\\ -(y_1z_2-y_2z_1)+(x_2z_1-x_1z_2)i & -(x_1y_2-x_2y_1)i\end{pmatrix}$$

& $$[\varphi(A),\varphi(B)] = \frac{1}{4}\left[\begin{pmatrix} z_1i & x_1+iy_1\\ -x_1+iy_1 & -z_1i\end{pmatrix}\begin{pmatrix} z_2i & x_2+iy_2\\ -x_2+iy_2 & -iz_2\end{pmatrix}-\begin{pmatrix} z_2i & x_2+iy_2\\ -x_2+iy_2 & -z_2i\end{pmatrix}\begin{pmatrix} z_1i & x_1+iy_1\\ -x_1+iy_1 & -z_1i\end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{pmatrix} (2x_1y_2-2x_2y_1)i & (2y_1z_2-2y_2z_1)+i(2x_2z_1-2x_1z_2)\\ -(2y_1z_2-2y_2z_1)+i(2x_2z_1-2x_1z_2) & -(2x_1y_2-2x_2y_1)i\end{pmatrix}$$ ✓

So this really is a Lie alg. isomorphism.

Homogeneous Spaces

Thm: Let H be a closed subgp of a Lie gp G & let G/H be the set of left cosets $\{gH : g\in G\}$. If $\pi: G\to G/H$ is the natural projection, then G/H has a unique mfd structure s.t.

① π is smooth &

② $\exists$ local sections of G/H: for $gH\in G/H$, $\exists$ nbhd U of gH & a smooth map $\sigma: W\to G$ s.t. $\pi\circ\sigma = \mathrm{id}|_W$.

The proof is a pain so we skip it. Manifolds of the form G/H are called homogeneous spaces.

More generally, suppose a Lie gp G acts on a mfd M; i.e., $\exists\ \eta: G\times M\to M$ s.t.
- η is smooth.
- $\eta(e,x)=x\ \ \forall x\in M$
- $\eta(gh,x)=\eta(g,\eta(h,x))\ \ \forall g,h\in G, x\in M$

(often write $g\cdot x$ for $\eta(g,x)$)

The action is called transitive if $\forall\, x, y \in M\ \exists\, g \in G$ s.t. $g \cdot x = y$.

For any $x \in M$, the isotropy subgroup of G at x is the group $G_x = \{g \in G : g \cdot x = x\}$.

(Notice, in particular, that we get the isotropy representation
$$\begin{aligned} G_x &\longrightarrow \mathrm{Aut}(T_x M) \\ g &\longmapsto dg_x\big|_{T_x M} \end{aligned}$$