# Math 670: Day 25

**Thm:** Let $G$ be a connected Lie gp & let $f, g: G \to H$ be Lie gp homomorphisms s.t. the Lie algebra homomorphisms

$df, dg: \mathfrak{g} \to \mathfrak{h}$ are identical. Then $f \equiv g$.

**Proof:** Since $df = dg$ as maps $\mathfrak{g} \to \mathfrak{h}$, their transposes $f^* = g^*$ as maps $\mathfrak{h}^* \to \mathfrak{g}^*$ where we identify $\mathfrak{g}^*$ & $\mathfrak{h}^*$ w/ the left-invt 1-forms on $G$ & $H$.

Now, suppose $\omega_1, \dots, \omega_n$ is a basis for the LI 1-forms on $H$. Then we know $f^*\omega_i = g^*\omega_i$ $\forall\, i = 1, \dots, n$.

By the uniqueness in the previous theorem, it will follow that $f = g$ if we can show that the ideal $\mathcal{I}$ on $G \times H$ generated by the

$$\mu_i = \pi_1^* f^* \omega_i - \pi_2^* \omega_i$$

is a differential ideal.

This is now a computation: since $\omega_1, \dots, \omega_n$ is a basis for the LI 1-forms on $H$, $\{\omega_i \wedge \omega_j : i < j\}$ is a basis for the LI 2-forms.

Now, for any $k \in \{1, \dots, n\}$, $d\omega_k$ is left-invt, so $\exists$ constants $c_{ijk}$ s.t. $d\omega_k = \sum_{i<j} c_{ijk}\, \omega_i \wedge \omega_j$.

But then

$$d\mu_k = d(\pi_1^* f^* \omega_k - \pi_2^* \omega_k) = \pi_1^* f^* d\omega_k - \pi_2^* d\omega_k = \pi_1^* f^* \sum c_{ijk}\, \omega_i \wedge \omega_j - \pi_2^* \sum c_{ijk}\, \omega_i \wedge \omega_j$$

$$= \sum c_{ijk} \left( \pi_1^* f^* \omega_i \wedge \pi_1^* f^* \omega_j - \pi_2^* \omega_i \wedge \pi_2^* \omega_j \right)$$

$$= \sum c_{ijk} \left[ (\pi_1^* f^* \omega_i - \pi_2^* \omega_i) \wedge \pi_1^* f^* \omega_j + \pi_2^* \omega_i \wedge (\pi_1^* f^* \omega_j - \pi_2^* \omega_j) \right]$$

which is certainly an element of $\mathcal{I}$, & we see that $\mathcal{I}$ is closed under exterior differentiation. ▨