Math 670: Day 24

Application of Frobenius

Suppose $f: G \to H$ is a Lie gp homomorphism. Then f^* pulls back LI fns on H to LI forms on G:

$$L_g^* f^* \omega = (f \circ L_g)^* \omega = (L_{f(g)} \circ f)^* \omega = f^* L_{f(g)}^* \omega = f^* \omega$$

In H, we can form a basis $\omega_1, \ldots, \omega_d$ of LI 1-forms so that at each point $h \in H$, $\{(\omega_i)_h\}$ is a basis for $T_h H$.

More generally, we can ask the following question: suppose we have mfds M^m & N^n & a basis $\omega_1, \ldots, \omega_n$ for the 1-forms on N, as well as a preferred collection $\alpha_1, \ldots, \alpha_n$ of 1-forms on M. Is there a (preferably unique) map $f: M \to N$ so that

$$f^* \omega_i = \alpha_i \quad \forall\ i = 1, \ldots, n?$$

If there were such an f, then let π_1, π_2 be the natural projections on $M \times N$ & define μ_i on $M \times N$ by

$$\mu_i := \pi_1^* f^* \omega_i - \pi_2^* \omega_i .$$

Let $\mathcal{I}$ be the ideal generated by the μ_i.

The **graph** of f is the submfd $g: M \to M \times N$ given by $g(x) = (x, f(x))$

I claim that (M, g) is an integral mfd of $\mathcal{I}$. To see this, we need to show $g^* \mu_i = 0 \ \forall\ i$.

But def $\pi_1 \circ g = id$ & $\pi_2 \circ g = f$, so

$$g^* \mu_i = g^*(\pi_1^* f^* \omega_i - \pi_2^* \omega_i) = (\pi_1 \circ g)^* f^* \omega_i - (\pi_2 \circ g)^* \omega_i = f^* \omega_i - f^* \omega_i = 0.$$

So this says: if f exists, its graph is an integral mfd of the ideal $\mathcal{I}$. So now we try to construct the graph (& hence f) from $\mathcal{I}$.

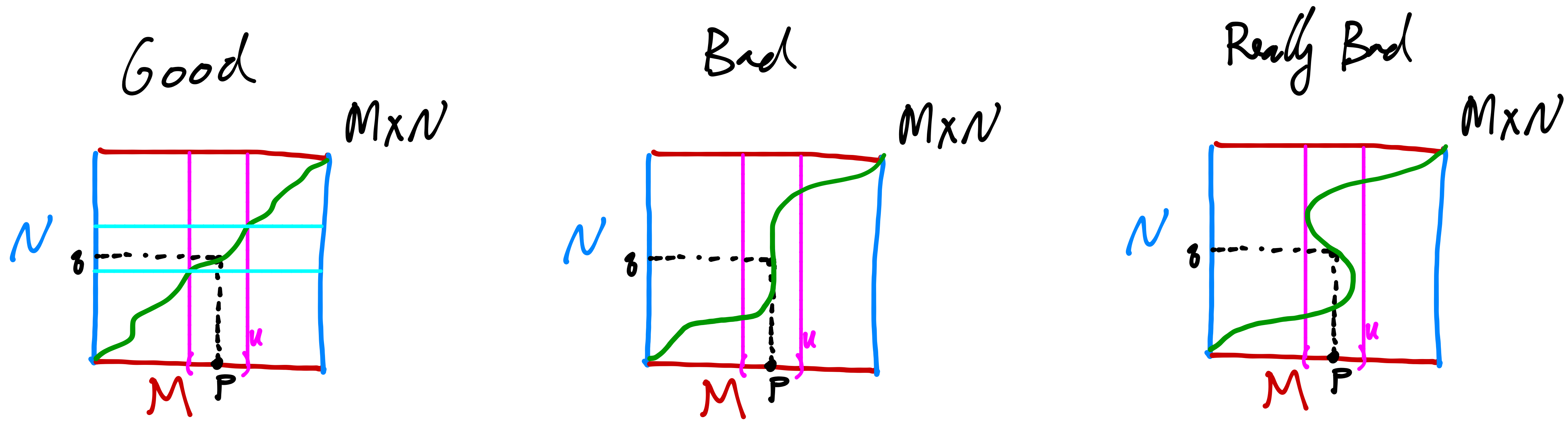
So now the idea is that we have mfds M^m & N^n, a basis $\omega_1, \ldots, \omega_n$ for the 1-forms on N, & a collection $\alpha_1, \ldots, \alpha_n$ of preferred 1-forms on M.

Define $\mu_i = \pi_1^* \alpha_i - \pi_2^* \omega_i \in \Omega^1(M \times N)$ and let $\mathcal{I}$ be the ideal generated by the μ_i.

Thm: If $\mathcal{I}$ is a differential ideal, then, given $p \in M$ & $q \in N$ $\exists$ nbhd U of p & a smooth map $f: U \to N$ s.t. $f(p) = q$ & $f^* \omega_i = \alpha_i \ \forall\ i = 1, \ldots, n$. Moreover, f is unique relative to the choice of U.

Proof: Since $\mathcal{I}$ is a differential ideal, Frobenius v.2 guarantees $\exists$ a maximal connected integral mfd S of $\mathcal{I}$ through the point $(p,q)\in M\times N$.

Now, the idea is to see that S is transverse to the M direction... if so, we can just define f to basically be $\pi_2\circ\pi_1^{-1}$:



I claim that $d\pi_1|_S$ is injective. To see this, let $(r,s)\in S$ & let $v\in T_{(r,s)}S$. If $d\pi_1(v)=0$ then

① $\mu_i(v)=0$ $\forall\, i=1,\dots,n$ since $\mathcal{I}$ consists of the forms which vanish on the tangent space to S.

② $\pi_1^*\alpha_i(v)=\alpha_i(d\pi_1 v)=\alpha_i(0)=0$.

But then ① & ② $\Rightarrow$ $\pi_2^*\omega_i(v)=0$ $\forall\, i=1,\dots,n$. Since the ω_i form a basis of 1-forms on M, this means $d\pi_2 v=0$.

But if $d\pi_1 v=0$ & $d\pi_2 v=0$, then $v=0$ since $T_{(r,s)}M\times N=T_rM\oplus T_sN$.

Since $(r,s)\in S$ was arbitrary, we see that $d\pi_1|_{TS}$ is injective, which means $\pi_1|_S: S\to M$ is a local diffeomorphism since $\dim S=(m+n)-n=m=\dim M$. Therefore, $\exists$ nbhd V of (p,q) in S & nbhd U of p s.t. $\pi_1|_V: V\to U$ is a diffeo.

So now define $f:U\to N$ by $f=\pi_2\circ(\pi_1|_V)^{-1}$. Then I claim that $f^*\omega_i=\alpha_i$ $\forall\, i=1,\dots,n$.

Let $u\in T_pM$. Then
$$f^*\omega_i(u)=(\pi_2\circ(\pi_1|_V)^{-1})^*\omega_i(u)=((\pi_1|_V)^{-1})^*\pi_2^*\omega_i(u)=\pi_2^*\omega_i(d(\pi_1|_V)^{-1}u)$$
$$\overset{\text{def of }\mu_i}{=}\pi_1^*\alpha_i(d(\pi_1|_V)^{-1}u)-\cancelto{0}{\mu_i(d(\pi_1|_V)^{-1}u)}$$
since $d(\pi_1|_V)^{-1}u\in T_{(r,s)}S$ & $\mathcal{I}=\mathcal{I}(S)$.
$$=\alpha_i(d\pi_1\, d(\pi_1|_V)^{-1}u)$$
$$=\alpha_i(d(\pi_1\circ(\pi_1|_V)^{-1})u)$$
$$=\alpha_i(u).$$
∎

Uniqueness is straightforward.