Math 670: Day 23

Remember that, given a distribution $\mathcal{D}$ on M^n, $\mathcal{I}(\mathcal{D}) \subseteq \Omega(M)$ is the ideal of forms which annihilate $\mathcal{D}$.

We showed that any point $x \in M$ is contained in a coord. chart (U, φ) w/ local coords $y_1, \dots, y_n$ s.t. $\mathcal{D} = \text{span}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k}\}$

If $\eta_1, \dots, \eta_n$ are the dual 1-forms to the coord. functions y_i, then $\eta_{k+1}, \dots, \eta_n$ are independent & generate $\mathcal{I}(\mathcal{D})$, which proves:

Lemma: For any k-dim'l dist. $\mathcal{D}$, the ideal $\mathcal{I}(\mathcal{D})$ is locally generated by $n-k$ independent 1-forms $\eta_{k+1}, \dots, \eta_n$.

OTOH...

Lemma: If $\mathcal{I} \subseteq \Omega(M)$ is an ideal locally generated by $n-k$ independent 1-forms, then there is a k-dim'l distribution $\mathcal{D}$ on M s.t. $\mathcal{I} = \mathcal{I}(\mathcal{D})$.

Proof: Let $\omega_{k+1}, \dots, \omega_n$ be the independent 1-forms generating $\mathcal{I}$ in a nbhd of $x \in M$. Then define $\mathcal{D}_x \subseteq T_xM$ to be the k-dim'l subspace annihilated by $\omega_{k+1}, \dots, \omega_n$. Then $\mathcal{D} = \bigcup_{x \in M} \mathcal{D}_x$ is the desired distribution. ∎

Frobenius Thm Version 2: A distribution $\mathcal{D}$ on M is involutive (& hence integrable) if & only if

$$d(\mathcal{I}(\mathcal{D})) \subseteq \mathcal{I}(\mathcal{D}).$$

Def: An ideal $\mathcal{I} \subseteq \Omega(M)$ is called a differential ideal if $d(\mathcal{I}) \subseteq \mathcal{I}$.

Proof of Frobenius v2: Locally choose $\omega_1, \dots, \omega_n$ that span $(T_xM)^* = \Lambda^1((T_xM)^*)$ so that $\omega_{k+1}, \dots, \omega_n$ generate $\mathcal{I}(\mathcal{D})$. Let $X_1, \dots, X_n$ be the dual vector fields, meaning that $\omega_i(X_j) = \delta_{ij}$. Then $X_1, \dots, X_k$ span $\mathcal{D}$ & so $\mathcal{D}$ is involutive $\Leftrightarrow [X_i, X_j] \in \mathcal{D}\ \forall 1 \leq i, j \leq k$

But now we get to use Cartan's Magic Formula: $L_X \omega = \iota_X d\omega + d\iota_X \omega$, which recall implied that

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

So, for $1 \leq i, j \leq k$ & $k+1 \leq p \leq n$, we have

$$d\omega_p(X_i, X_j) = X_i(\cancelto{0}{\omega_p(X_j)}) - X_j(\cancelto{0}{\omega_p(X_i)}) - \omega_p([X_i, X_j]) = -\omega_p([X_i, X_j]) = 0 \Leftrightarrow \mathcal{D} \text{ is involutive}$$

since $\omega_k \in \mathcal{I}(\mathcal{D})$. So $\mathcal{I}(\mathcal{D})$ being a differential ideal & $\mathcal{D}$ being involutive are equivalent. ∎

Ex: Remember our LI vector fields X, Y, Z on S^3, where $X_p = ip$, $Y_p = jp$, $Z_p = kp$ $\forall p \in S^3$, & the dual 1-forms ξ, η, ζ. We can define the 2-dim'l distribution $\mathcal{D}$ on S^3 to be $\mathcal{D} = \text{span}\{Y, Z\}$. Then by the Lemma we know $\mathcal{I}(\mathcal{D})$ is spanned by a 1-form, which is obviously ξ.

Now, we showed before that $[Y,Z] = -2X$, so $\mathcal{D}$ is not involutive & hence, by Frobenius, not integrable even locally.

We could also use Frobenius v.2 by checking that $\mathcal{I}(\mathcal{D})$ is not a differential ideal.

Indeed, we showed before that $d\xi = 2\eta\wedge\zeta \notin \mathcal{I}(\mathcal{D})$, since $2\eta\wedge\zeta(Y,Z) = 2 \neq 0$.