Math 670: Day 22

Frobenius Thm: Let $\mathcal{D}$ be a k-dim'l involutive distribution on M^n. Then for each $x \in M$ there is an integral mfd of $\mathcal{D}$ through x.

Idea behind the theorem: Suppose $\mathcal{D}$ is (locally) spanned by the vector fields $V_1, \ldots, V_k$. Then (remembering that the V_i are derivations & hence 1st order differential operators on M), the collection $\{V_1, \ldots, V_k\}$ determines the first order system of PDEs

$$V_1 u = 0, \ldots, V_k u = 0.$$

If $u: M \to \mathbb{R}$ is a solution, then the level sets of u are tangent to $\langle V_1, \ldots, V_k \rangle$. Frobenius' question was: when are there solutions

$u_1, \ldots, u_{n-k}$ s.t. $\{\nabla u_1, \ldots, \nabla u_{n-k}\}$ are linearly independent?

When there exist such solutions, the level sets of the map $(u_1, \ldots, u_{n-k}): M \to \mathbb{R}^{n-k}$ will necessarily be integral manifolds of $\mathcal{D}$.

Proof of Frobenius: I want to show that $\forall x \in M$ $\exists$ coord. chart (U, φ) s.t.

① $\vec{0} \in U \subseteq \mathbb{R}^n$

② $\varphi(\vec{0}) = x$

③ For $\vec{a} = \{a_1, \ldots, a_n\} \in U$, the set $\{p \in M : \varphi^{-1}(p)_{k+1} = a_{k+1}, \ldots, \varphi^{-1}(p)_n = a_n\}$ is an integral mfd of $\mathcal{D}$.

Since this is local, may as well assume $M \subseteq \mathbb{R}^n$, $x = \vec{0}$ & $\mathcal{D}(x) = \mathcal{D}(\vec{0}) = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \rangle$.

Let $\pi: \mathbb{R}^n \to \mathbb{R}^k$ be the projection, so $d\pi_0|_{\mathcal{D}}: \mathcal{D}(\vec{0}) \to T_{\vec{0}}\mathbb{R}^k$ is an isomorphism & hence $d\pi$ is injective on some nbhd U of $\vec{0}$.

Therefore, in U $\exists!$ $V_1, \ldots, V_k \in \mathcal{D}$ s.t. $d\pi_q V_i = \frac{\partial}{\partial x_i}\Big|_{\pi(q)}$ $\forall q \in U$ & $\forall i \in \{1, \ldots, k\}$.

Since $\mathcal{D}$ is involutive, $[V_i, V_j] \in \mathcal{D}$ $\forall$ $1 \le i, j \le k$. OTOH,

$$d\pi[V_i, V_j] = [d\pi V_i, d\pi V_j] = [\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}] = 0, \text{ so } [V_i, V_j] = 0 \text{ since } d\pi \text{ an isomorphism.}$$

But then we can find coords $(y_1, \ldots, y_n)$ s.t. $V_i = \frac{\partial}{\partial y_i}$ on U. But then the sets

$$\{y \in U : y_{k+1} = a_{k+1}, \ldots, y_n = a_n\} \text{ are integral mfds of } \mathcal{D}.$$ ∎

This is a local result, but it turns out that the local pieces glue together nicely:

Global Frobenius Thm: Let $\mathcal{D}$ be an involutive distribution on M. Then M is *foliated* by a (disconnected) integral mfd for $\mathcal{D}$.

The components of this foliation are called the *maximal integral manifolds* for $\mathcal{D}$.

Differential Ideals

Suppose $\mathcal{D}$ is a k-dim'l distribution on M^n & define $\mathcal{I}(\mathcal{D}) \subseteq \Omega(M)$ to be the ring of differential forms which annihilate $\mathcal{D}$:

$$\mathcal{I}(\mathcal{D}) := \bigcup_p \{\omega \in \Omega^p(M) : \omega(X_1, .., X_p) = 0 \quad \forall X_1, .., X_p \in \mathcal{D}\}$$

If $\omega_1, \omega_2 \in \mathcal{I}(\mathcal{D})$, then $\omega_1 + \omega_2 \in \mathcal{I}(\mathcal{D})$ & $\omega_1 \wedge \omega_2 \in \mathcal{I}(\mathcal{D})$, so this really is a ring.

In fact, if $\omega \in \mathcal{I}(\mathcal{D}) \cap \Omega^p(M)$ & $\eta \in \Omega^q(M)$, then

$$\omega \wedge \eta\,(X_1, .., X_{p+q}) = \sum_{\sigma \in S_{(p,q)}} \operatorname{sgn}(\sigma)\, \omega(X_{\sigma(1)}, .., X_{\sigma(p)})\, \eta(X_{\sigma(p+1)}, .., X_{\sigma(q)}) = 0,$$

so $\omega \wedge \eta \in \mathcal{I}(\mathcal{D})$ & we see that $\mathcal{I}(\mathcal{D})$ is an ideal.