Math 670: Day 21

Goal: **Thm:** Let G & H be Lie gps w/ Lie algebras $\mathfrak{g}$ & $\mathfrak{h}$, respectively, & with G simply-connected. Let $\psi: \mathfrak{g} \to \mathfrak{h}$ be a Lie alg. homomorphism. Then $\exists!$ homomorphism $\varphi: G \to H$ s.t. $d\varphi = \psi$.

Cor: If simply connected Lie gps G & H have isomorphic Lie algebras, then G & H are isomorphic.

First step:

Prop: Let G & H be Lie gps & let $\varphi: G \to H$ a homomorphism. Then $d\varphi: \mathfrak{g} \to \mathfrak{h}$ is a Lie alg. homomorphism, where $d\varphi X$ really means we extend $d\varphi X$ to a LI v.f. on all of H.

Proof: If $X \in \mathfrak{g}$, then $dL_{\varphi(g)} d\varphi\, X = d(L_{\varphi(g)} \circ \varphi)\, X = d(\varphi \circ L_g) X = d\varphi\, dL_g X = d\varphi X$ since X is LI, so $d\varphi X$ is LI on the image of φ & hence extends to an elmt of $\mathfrak{h}$.

Now, from an earlier lemma we know that, for $X, Y \in \mathfrak{g}$, $d\varphi_g [X,Y] = [d\varphi X, d\varphi Y]_{\varphi(g)}$, so $d\varphi$ is a Lie alg. homomorphism. ☒

To get to the theorem, we need to talk a bit about distributions.

Def: A d-dim'l distribution on a mfd M^n is a smooth choice of d-dim'l subspace $\mathcal{D}(x) \subseteq T_x M$ $\forall x \in M$.

In other words, $\mathcal{D}$ is a sub-bundle of the tangent bundle TM.

A v.f. $X \in \mathfrak{X}(M)$ lies in $\mathcal{D}$ if $X_p \in \mathcal{D}(p)$ $\forall p \in M$.

The distribution is involutive if $[X,Y] \in \mathcal{D}$ $\forall X, Y \in \mathcal{D}$.

Ex: ① Let $\mathcal{D} = \text{span}\{\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}}\}$ for some $1 \le i_1 < i_2 < \ldots < i_k \le n$. Then $\mathcal{D}$ is a k-dim'l dist. on $\mathbb{R}^n$.

Moreover, for any $1 \le p, q \le k$, $[\frac{\partial}{\partial x_{i_p}}, \frac{\partial}{\partial x_{i_q}}] = 0$ since mixed partials commute, so $\mathcal{D}$ is involutive.

② Let $\xi = \text{span}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\} = \ker(dz - y\,dx)$ be a 2-dim'l dist. on $\mathbb{R}^3$. Then

$$[\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial z}](f) = \tfrac{\partial}{\partial y}(\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial z})(f) - (\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial z})(\tfrac{\partial}{\partial y})(f) = \ldots = \frac{\partial f}{\partial z},$$ so

$[\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}] = \frac{\partial}{\partial z} \notin \xi$, so ξ is not involutive.

ξ is called the standard contact structure on $\mathbb{R}^3$.

Def: The image of a smooth embedding $i: N \to M$ is an *integral manifold* of a distribution $\mathcal{D}$ on M if

$$di(T_pN) = \mathcal{D}(i(p)) \quad \forall\ p \in N.$$

Prop: If $\mathcal{D}$ is a smooth distribution on M s.t. at each $x \in M$ there is an integral mfd of $\mathcal{D}$ passing thru x, then $\mathcal{D}$ is involutive.

Proof: If $V, W \in \mathcal{D}$, then at each $x = i(p) \in i(N)$, $V_x \in di_p V'$ & $W' = di_p W'$ we have

$$[V', W']_p \in T_pN \ \text{\& hence}\ [V,W]_x = [di_p V', di_p W']_x = (di_p[V',W'])_x \in di_p(T_pN) = \mathcal{D}(x).$$ So $\mathcal{D}$ is involutive ∎

The much more amazing fact is that the converse is true. This is called the Frobenius Theorem:

Frobenius Thm: Let $\mathcal{D}$ be a k-dim'l involutive distribution on M^n. Then for each $x \in M$ there is an integral mfd of $\mathcal{D}$ through x.