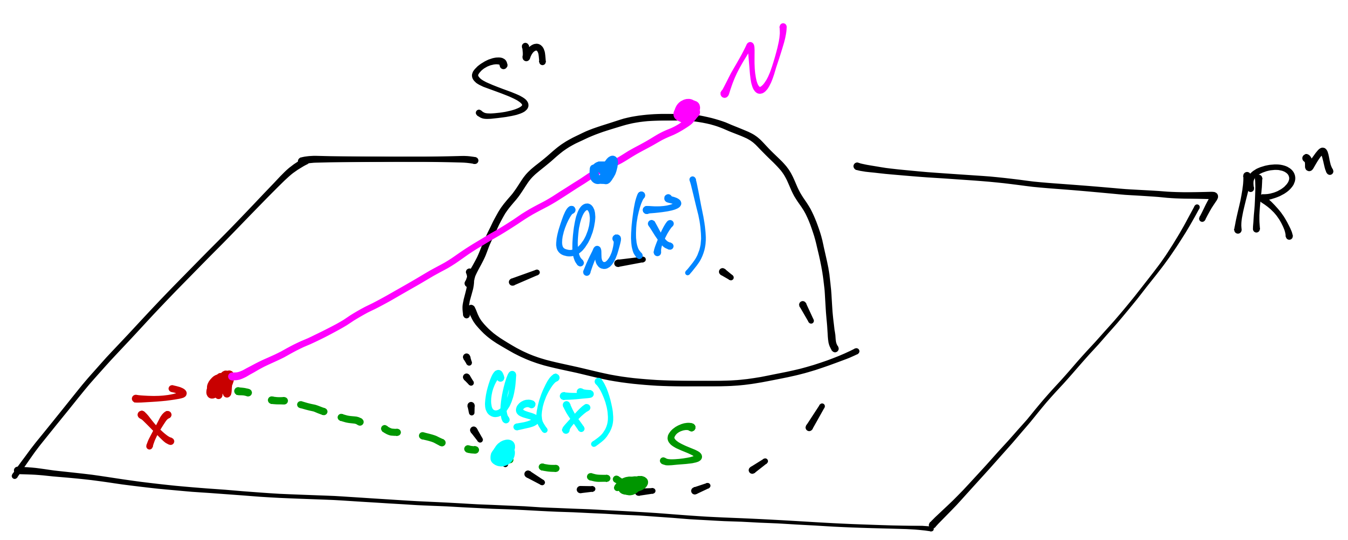
Math 670: Day 2

Ex: • S^n, the unit sphere in $\mathbb{R}^{n+1}$. The maximal collection containing $\{(\mathbb{R}^n, \varphi_N), (\mathbb{R}^n, \varphi_S)\}$ makes S^n into an n-mfd,

where φ_N & φ_S are inverse stereographic projection from the north & south poles, respectively.

Specifically, $\varphi_N(x_1, \ldots, x_n) = \left(\frac{2x_1}{1+x_1^2+\ldots+x_{n-1}^2}, \ldots, \frac{2x_{n-1}}{1+x_1^2+\ldots+x_{n-1}^2}, \frac{-1+x_1^2+\ldots+x_n^2}{1+x_1^2+\ldots+x_n^2} \right)$

& $\quad \varphi_S(x_1, \ldots, x_n) = \left(\frac{2x_1}{1+x_1^2+\ldots+x_n^2}, \ldots, \frac{2x_{n-1}}{1+x_1^2+\ldots+x_{n-1}^2}, \frac{1-x_1^2-\ldots-x_{n-1}^2}{1+x_1^2+\ldots+x_{n-1}^2} \right)$



Clearly ① is satisfied. For ②, notice that $\varphi_N(\mathbb{R}^n) \cap \varphi_S(\mathbb{R}^n) = S^n - \{N, S\}$. Now,

$\varphi_N^{-1}(S^n - \{N,S\}) = \mathbb{R}^n$, which is trivially open, & likewise $\varphi_S^{-1}(S^n - \{N,S\}) = \mathbb{R}^n$.

Moreover, $\varphi_S^{-1} \circ \varphi_N : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Can check that $\varphi_S^{-1}(y_1, \ldots, y_{n+1}) = \frac{1}{1+y_{n+1}}(y_1, \ldots, y_n)$, so

$(\varphi_S^{-1} \circ \varphi_N)(x_1, \ldots, x_n) = \frac{1}{x_1^2+\ldots+x_n^2}(x_1, \ldots, x_n)$, which is clearly smooth.

A similar calculation works for $\varphi_N^{-1} \circ \varphi_S$.

Def: Let M^m, N^n be mfds. A cont. map $f: M \to N$ is *differentiable* at $p \in M$ if, given a coordinate chart

$\psi: V \subseteq \mathbb{R}^n \to N$ containing $f(p)$, $\exists$ chart $\varphi: U \subseteq \mathbb{R}^m \to M$ containing p s.t. $f(\varphi(U)) \subseteq \psi(V)$ and

$$\psi^{-1} \circ f \circ \varphi : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$$

is differentiable at $\varphi^{-1}(p)$. The map f is differentiable on an open set in M if it is diff'ble at every point in that set.

Ex: $f: S^n \to S^n$ given by $f(\vec{y}) = -\vec{y}$. Then $\varphi_S^{-1} \circ f \circ \varphi_N(\vec{x}) = -\vec{x}$, which is obviously diff'ble everywhere (this doesn't cover the case $\vec{y} = N$, of course)

Tangent Vectors

To define tangent vectors on a manifold, we're going to work by analogy with a way of thinking about vectors in $\mathbb{R}^n$ that's slightly different than what you may be used to. In words, we'll identify a vector $v \in \mathbb{R}^n$ w/ the operator on differentiable functions which gives the directional derivative w.r.t. v of a function.

More precisely, if $\alpha: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is smooth (i.e., a smooth curve) w/ $\alpha(0) = p$, then write

$$\alpha(t) = (x_1(t), \ldots, x_n(t)), \quad t \in (-\varepsilon, \varepsilon), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then $\alpha'(0) = (x_1'(0), \ldots, x_n'(0)) = v \in \mathbb{R}^n$. Suppose f is diff'ble in a nbhd of p. Then, restricting f to the image of α, the directional derivative of f in the direction of v is:

$$\left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0} = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p \left.\frac{dx_i}{dt}\right|_{t=0} = \left(\sum_i x_i'(0) \frac{\partial}{\partial x_i}\right) f.$$

On the right we've now written the directional derivative as an operator acting on f. Moreover, this operator depends uniquely on v & is a linear derivation, meaning that ① $v(f + \lambda g) = v(f) + \lambda v(g)$ $\forall f, g$ diff'ble in a nbhd of p & all $\lambda \in \mathbb{R}$

② $v(fg) = f(p) v(g) + g(p) v(f)$

Def: Let M^n be a mfd. A smooth function $\alpha: (-\varepsilon, \varepsilon) \to M$ is a (smooth) curve in M. Suppose $\alpha(0) = p \in M$ & let $\mathcal{D}$ be the set of functions on M that are differentiable at p. The tangent vector to the curve α at $t=0$ is a function $\alpha'(0): \mathcal{D} \to \mathbb{R}$ given by $\alpha'(0) f = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}$.

A tangent vector at p is the tangent vector at $t=0$ of some curve $\alpha: (-\varepsilon, \varepsilon) \to M$ w/ $\alpha(0) = p$.

The set of all tangent vectors at p is the tangent space to M at p, denoted T_pM.