Math 670: Day 19-20

Goal is to prove

Thm: On a cpct, connected Lie gp (CCLG), the algebra of bi-invariant forms is isomorphic (in the obvious way) to the de Rham cohomology algebra.

This is an immediate consequence of the following 2 propositions:

Prop 1: On a CCLG, every closed form is cohomologous to a bi-invt form.

Prop 2: On a CCLG, a bi-invt form which is exact must be identically zero.

We build up:

Lemma 1: Every bi-invt form on a Lie gp is closed.

Proof: Let G be a Lie gp & let $inv: G \to G$ be given by $inv(g) = g^{-1}$.

If $\omega \in \Omega^k(G)$, then $inv^* \omega_e = \begin{cases} \omega_e & \text{if } k \text{ even} \\ -\omega_e & \text{if } k \text{ odd} \end{cases}$

since the differential of inversion at $e \in G$ is $-I$.

Now, since $inv \circ L_{g^{-1}} = R_g \circ inv$ or $inv = R_g \circ inv \circ L_g$ then, for bi-invt $\omega \in \Omega^k(G)$,

$$inv^* \omega_g = (R_g \circ inv \circ L_g)^* \omega_g = L_g^* inv^* R_g^* \omega_g = L_g^* inv^* \omega_e = L_g^* (-1)^k \omega_e = (-1)^k \omega_{g^{-1}} \quad \forall g \in G,$$

So $inv^* \omega = (-1)^k \omega$. Now, on one hand

$$inv^* d\omega = d\, inv^* \omega = d(-1)^k \omega = (-1)^k d\omega$$

but OTOH, since $d\omega$ is bi-invt, we know $inv^* d\omega = (-1)^{k+1} d\omega$, so we must have $d\omega = 0$. ▨

Def: If M^n a mfd, then a nowhere-vanishing n-form ω on M is called a volume form on M.

Note: Having a volume form is equivalent to being orientable.

Every Lie gp G has a left-invt volume form: let $\omega_e \in \Lambda^n(\mathfrak{g}^*)$ be nonzero & define $\omega_g = L_{g^{-1}}^* \omega_e \quad \forall g \in G$.

But this form may not be bi-invt.

Ex: Show that there are no bi-invt 2-forms on $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \neq 0 \right\}$: the form $\xi \wedge \eta$ is LI but not BI.

Thm: A CCLG G admits a (unique up to constant multiples) bi-inv't volume form

Note: The measure induced by this volume form is Haar measure.

Proof: Let ω be a LI volume form, so $L_g^*\omega=\omega\ \forall g\in G$.

Since left & right multiplication commute, $R_g^*\omega$ must be LI & hence must be some nonzero multiple of ω, with the multiplier depending on g:

$$R_g^*\omega=\varphi(g)\,\omega$$

Since $R_g\circ R_h=R_{hg}$, we have

$$\varphi(hg)=\varphi(g)\varphi(h),$$

so $\varphi: G\to\mathbb{R}^*$ is a homomorphism, & hence $\mathrm{im}(\varphi)$ is a subgp of $\mathbb{R}^*$.

But since G is compact & connected, $\mathrm{im}(\varphi)$ must be the unique cpct, connected subgp of $\mathbb{R}^*$, namely $\{1\}$.

But then $R_g^*\omega=\varphi(g)\omega=\omega\ \forall g\in G$, so the LI volume form ω is also RI & hence BI. ☒

Now we're ready:

Proof of Prop 1: Let $\omega\in\Omega^k(G)$ be closed: $d\omega=0$. Since G is connected, $L_g: G\to G$ & $R_g: G\to G$ are homotopic to the identity, so $L_g^*\omega-\omega$ is cohomologous to $id^*\omega-\omega=\omega-\omega=0$, so $L_g^*\omega$ is cohomologous to $\omega\ \forall g\in G$; likewise, $R_g^*\omega$ is cohomologous to ω.

Now, G is compact, so it has a bi-inv't volume form $d\mathrm{vol}$. Define the average

$$\overline{\omega}:=\frac{1}{(\mathrm{vol}\,G)^2}\int_{G\times G}L_g^*R_h^*\omega\, d\mathrm{vol}_g\, d\mathrm{vol}_h.$$

Now, $d\,L_g^*R_h^*\omega=L_g^*\,d\,R_h^*=L_g^*R_h^*\,d\omega=0$, so $\overline{\omega}$ is closed. Also, $L_g^*R_h^*\omega$ is cohomologous to ω, so $\overline{\omega}$ & ω are cohomologous.

Finally, $\overline{\omega}$ is bi-inv't. For example,

$$L_a^*\overline{\omega}=\frac{1}{(\mathrm{vol}\,G)^2}\int_{G\times G}L_a^*L_g^*R_h^*\omega\, d\mathrm{vol}_g\, d\mathrm{vol}_h=\frac{1}{(\mathrm{vol}\,G)^2}\int_{G\times G}L_{ga}^*R_h^*\omega\, d\mathrm{vol}_g\, d\mathrm{vol}_h$$

right-invariance of $d\mathrm{vol}$

$$=\frac{1}{(\mathrm{vol}\,G)^2}\int_{G\times G}L_{ga}^*R_h^*\omega\, d\mathrm{vol}_{ga}\, d\mathrm{vol}_h$$

$$=\frac{1}{(\mathrm{vol}\,G)^2}\int_{G\times G}L_g^*R_h^*\, d\mathrm{vol}_g\, d\mathrm{vol}_h$$

$$=\overline{\omega},$$

so $\overline{\omega}$ is LI. A similar computation shows $\overline{\omega}$ is RI & hence BI ☒

Remarks: ① If ω already BI, then $\overline{\omega} = \omega$.

② The averaging process commutes w/ exterior derivative: $\overline{d\eta} = d\overline{\eta}$.

Proof of Prop 2: Suppose $\omega \in \Omega^k(G)$ is BI & exact, so $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(G)$ w/ no particular invariance properties.

Then $\omega = \overline{\omega} = \overline{d\eta} = d\overline{\eta}$ where $\overline{\eta}$ is bi-invt. But by our earlier lemma, all bi-invt forms are closed, so $\omega = d\overline{\eta} = 0$. ∎

And now the proof of the theorem is simple:

Proof of Thm: By Prop 1, every closed form is cohomologous to a BI form, so we'll have at least one BI form in each de Rham cohomology class.

If there were more than one BI form in a given class, their difference would be BI & exact & hence zero by Prop 2. ∎