Math 670: Day 18

An algorithm for finding LI 1-forms

Suppose G is a subgp of $GL(n,\mathbb{R})$. We'll demonstrate the method w/ our gp G of affine transf'ns of $\mathbb{R}$ from before.

Write down a typical elt. of G in local coords: $B = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$

Then $B^{-1} = \begin{pmatrix} \frac{1}{x} & -y \\ 0 & 1 \end{pmatrix}$ & $dB = \begin{pmatrix} dx & dy \\ 0 & 0 \end{pmatrix}$.

And now $B^{-1}dB = \begin{pmatrix} \frac{1}{x}dx & \frac{1}{x}dy \\ 0 & 0 \end{pmatrix}$ & we recognize the entries of $B^{-1}dB$ as the LI 1-forms.

Why does this work? If $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is a typical elt of G, then $L_A(B) = AB$ & $L_A^* dB = A\,dB$ since for $v \in \mathfrak{X}(G)$,

$$L_A^* dB(v) = dB(dL_A v) = dB(Av) = A\,dB(v).$$

Hence $L_A^* B^{-1}dB = L_A^* \operatorname{inv}(B)\, dB = (\operatorname{inv} \circ L_A)(B)\, L_A^* dB = (AB)^{-1} A\,dB = B^{-1}A^{-1}A\,dB = B^{-1}dB$, so

$B^{-1}dB$ is a left-invt 1-form.

An algorithm for computing the exterior derivative of the LI 1-forms

Let G be a matrix gp & let B be the matrix representing a typical elt. in local coords.

Then, as before, consider $B^{-1}dB$, whose entries are LI 1-forms.

Writing $\Omega = B^{-1}dB$ or, equivalently, $dB = B\Omega$, we have

$$0 = d^2 B = d(dB) = d(B\Omega) = dB \wedge \Omega + B\,d\Omega = B\Omega \wedge \Omega + B\,d\Omega = B(\Omega \wedge \Omega + d\Omega),$$

yielding the Maurer-Cartan equation $d\Omega + \Omega \wedge \Omega = 0$.

So on our gp G of affine transf'ns of $\mathbb{R}$, we have $\Omega = B^{-1}dB = \begin{pmatrix} \frac{1}{x}dx & \frac{1}{x}dy \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \xi & \eta \\ 0 & 0 \end{pmatrix}$ & so

$\begin{pmatrix} d\xi & d\eta \\ 0 & 0 \end{pmatrix} = d\Omega = -\Omega \wedge \Omega = \begin{pmatrix} 0 & -\xi \wedge \eta \\ 0 & 0 \end{pmatrix}$, meaning that $d\xi = 0$ & $d\eta = -\xi \wedge \eta$, agreeing w/ our earlier calculation.

Bi-invariant forms

Let G be a Lie gp, let $L_g: G\to G$ & $R_g: G\to G$ denote left & right multiplication respectively by $g\in G$. Then $\omega\in\Omega^k(G)$

is called bi-invariant if $L_g^*\omega=\omega$ & $R_g^*\omega=\omega$ $\forall g\in G$.

Ex: On an abelian Lie gp, left inv $\Rightarrow$ right inv, so all LI forms are bi-inv.

How to find BI forms?

Let $C_g: G\to G$ denote conjugation by g: $C_g(h)=ghg^{-1}$.

If $\omega\in\Omega^k(G)$ is LI w/ value ω_e at the identity, & suppose ω_e is conjugation-inv: $C_g^*\omega_e=\omega_e$ $\forall g\in G$. Then ω must be RI since $R_g=C_{g^{-1}}\circ L_g$, so $\omega_g=L_{g^{-1}}^*\omega_e=L_{g^{-1}}^*C_g^*\omega_e=(C_{g^{-1}}\circ L_g)^*\omega_e=R_{g^{-1}}^*\omega_e$ $\forall g\in G$ & hence

$$R_h^*\omega_g=R_h^*R_{g^{-1}}^*\omega_e=(R_{g^{-1}}\circ R_h)^*\omega_e=R_{hg^{-1}}^*\omega_e=\omega_{gh^{-1}}\quad\text{for all } g,h\in G.$$

In other words, conjugation-inv forms on $T_eG=\mathfrak{g}$ produce bi-inv forms on G.
Consider our gp G of affine transformations of the line again, w elements $\begin{pmatrix}x&y\\0&1\end{pmatrix}$ w/ $x\neq 0$.

In any Lie gp, C_g is a diffeo. taking e to itself, so the differential of C_g maps $T_eG=\mathfrak{g}$ to itself.

This map is denoted $Ad_g:\mathfrak{g}\to\mathfrak{g}$ & called the adjoint action of G on $\mathfrak{g}$.

Now, if $X=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $Y=\begin{pmatrix}0&1\\0&0\end{pmatrix}\in\mathfrak{g}$ & $\begin{pmatrix}a&b\\0&1\end{pmatrix}\in G$, then

$$Ad_{\begin{pmatrix}a&b\\0&1\end{pmatrix}}:\begin{pmatrix}1&0\\0&0\end{pmatrix}\mapsto\begin{pmatrix}a&b\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}a&b\\0&1\end{pmatrix}^{-1}=\begin{pmatrix}a&b\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}\frac{1}{a}&-b/a\\0&1\end{pmatrix}=\begin{pmatrix}1&-b\\0&0\end{pmatrix}$$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\mapsto\begin{pmatrix}a&b\\0&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}a&b\\0&1\end{pmatrix}^{-1}=\begin{pmatrix}a&b\\0&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\frac{1}{a}&-b/a\\0&1\end{pmatrix}=\begin{pmatrix}0&a\\0&0\end{pmatrix},$$

so $Ad_{\begin{pmatrix}a&b\\0&1\end{pmatrix}}: X\mapsto X-bY$
$\qquad Y\mapsto aY$

& we see that the matrix of $Ad_{\begin{pmatrix}a&b\\0&1\end{pmatrix}}$ w.r.t. this basis is $\begin{pmatrix}1&0\\-b&a\end{pmatrix}$, w/ eigenvalues 1 & a & corresponding eigenvectors $\begin{pmatrix}a-1\\b\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$.
Of course, the transpose action $Ad^*_{\begin{pmatrix}a&b\\0&1\end{pmatrix}}:\mathfrak{g}^*\to\mathfrak{g}^*$ has the matrix $\begin{pmatrix}1&-b\\0&a\end{pmatrix}$ w.r.t. the dual basis ξ,η, so maps

$$\xi\mapsto\xi \quad \& \quad \eta\mapsto -b\xi+a\eta$$

& hence maps $\xi \wedge \eta \mapsto a\,\xi \wedge \eta$.

Thus, ξ is the only bi-invt 1-form on G & there are no bi-invt 2-forms.

A Lie gp is called unimodular if it has a bi-invt form of top dimension, so we see that this gp G is not unimodular.

OTOH, we will see that every compact Lie gp is unimodular.