Math 670: Day 17

Thm: Suppose G is a Lie group & $\mathfrak{g}$ is the set of left-inv't vector fields on G. Then

(i) $\mathfrak{g}$ is a real vector space & the evaluation map $\varphi_e : \mathfrak{g} \to T_eG$ given by $\varphi_e(X) = X_e$ is a v.s. isomorphism,

so $\dim \mathfrak{g} = \dim T_eG = \dim G$.

(ii) Elements of $\mathfrak{g}$ are smooth vector fields.

(iii) For $X, Y \in \mathfrak{g}$, $[X,Y] \in \mathfrak{g}$, where $[\cdot,\cdot]$ is the usual Lie bracket of vector fields

(iv) $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra.

Proof: (i) φ_e is obviously linear. It's surjective since, if $v \in T_eG$, then we can define a vector field X on G by $X_g = dL_g(v)$. Then

$X \in \mathfrak{g}$ since $$(dL_h)_g(X) = dL_h\, dL_g\, v = dL_{hg}(v) = X_{hg}.$$

φ_e is also injective: if $\varphi_e(X) = \varphi_e(Y)$, then $\forall\, g \in G$,

$$X_g \overset{\text{L.I.}}{=} (dL_g)_e(X_e) = (dL_g)_e\, \varphi_e(X) = (dL_g)_e(\varphi_e(Y)) = (dL_g)_e(Y_e) \overset{\text{L.I.}}{=} Y_g,$$

so we see that $X = Y$.

(ii) is straightforward, but annoying, so we skip it.

(iii) is a consequence of the following lemma, & (iv) is immediate given that the Lie bracket of v.f.'s is alternating & satisfies the Jacobi identity. $\boxtimes$

Lemma: Suppose $f: M \to N$ is smooth, $X_M, Y_M \in \mathfrak{X}(M)$, $X_N, Y_N \in \mathfrak{X}(N)$ s.t. $df_p(X_M)_p = (X_N)_{f(p)}$ & $df_p(Y_M)_p = (Y_N)_{f(p)}$. Then

$$df_p[X_M, Y_M] = [X_N, Y_N]_{f(p)}.$$

In the theorem, we need the special case $M = N = G$ & $f = L_g$.

Proof of Lemma: Suppose $\psi \in C^\infty(N)$. Then we need $(df_p[X_M,Y_M])(\psi) = [X_N, Y_N]_{f(p)}(\psi)$.

Now, $(df_p[X_M,Y_M]_p)(\varphi) = [X_M,Y_M]_p(\varphi\circ f) = (X_M)_p(Y_M(\varphi\circ f)) - (Y_M)_p(X_M(\varphi\circ f))$

$$= (X_M)_p\big((df_p Y_M)(\varphi)\big) - (Y_M)_p\big((df_p X_M)(\varphi)\big)$$

$$\overset{\text{by hypothesis}}{=} (X_M)_p\big((Y_N)_{f(p)}(\varphi)\big) - (Y_M)_p\big((X_N)_{f(p)}(\varphi)\big)$$

$$= (X_M)_p\big(Y_N(\varphi)\circ f\big) - (Y_M)_p\big(X_N(\varphi)\circ f\big)$$

$$= (df_p X_M)(Y_N(\varphi)) - (df_p Y_M)(X_N(\varphi))$$

$$\overset{\text{by hypothesis}}{=} (X_N)_{f(p)}(Y_N(\varphi)) - (Y_N)_{f(p)}(X_N(\varphi))$$

$$= [X_N,Y_N]_{f(p)}(\varphi). \qquad \blacksquare$$

Thus, we can always associate the Lie algebra $\mathfrak{g}$ of L.I. v.f.'s to any Lie gp G; this is called the Lie algebra of G, & sometimes denoted $\mathrm{Lie}(G)$.

Ex: Consider the set of affine maps $t \mapsto at+b$ of the real line. Clearly, if $a \neq 0$ then such a map is invertible & the composition of two affine maps is affine, so the set of invertible affine maps of $\mathbb{R}$ forms a Lie gp. G.

Now, consider the matrix eqn $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} xt+y \\ 1 \end{bmatrix}$

So, identifying $\mathbb{R}$ w/ $\left\{\begin{pmatrix} t \\ 1 \end{pmatrix} : t \in \mathbb{R}\right\} \subseteq \mathbb{R}^2$, we can represent G as the subgp of $GL(2,\mathbb{R})$ given by $\left\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \neq 0\right\}$.

Now, $T_I G$ is generated by $X := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ & $Y := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ & by abuse of notation we can either think of X & Y as these specific matrices or as the corresponding L.I. vf's on G.

Exercise: As LI vfs, $X = x\frac{\partial}{\partial x}$ & $Y = x\frac{\partial}{\partial y}$.

Exercise: The Lie bracket $[X,Y] = XY - YX = Y$

The Lie algebra $\mathfrak{g}$ turns out to be the only nonabelian 2-dim'l Lie alg.

It is solvable, since the derived series terminates: $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}] = \langle Y \rangle$ & $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}] = 0$

It is not nilpotent, since the lower central series gets fixed forever: $\mathfrak{g}_{(1)} = [\mathfrak{g},\mathfrak{g}] = \langle Y \rangle$, $\mathfrak{g}_{(2)} = [\mathfrak{g},\mathfrak{g}_{(1)}] = \mathfrak{g}_{(1)}$, etc.

If ξ & η are the dual LI 1-forms, then
$$d\xi(X,Y) = X(\xi(Y)) - Y(\xi(X)) - \xi([X,Y]) = X(0) - Y(1) - \xi(Y) = 0$$
$$d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = X(1) - Y(0) - \eta(Y) = -1$$

So $d\xi = 0$ & $d\eta = -\xi \wedge \eta$

Exercise: $\xi = \frac{1}{x} dx$, $\eta = \frac{1}{x} dy$.

Notice that, in particular, this is going to imply that $H^0(G,\mathbb{R}) \cong \mathbb{R}$, $H^1(G,\mathbb{R}) \cong \mathbb{R}$, $H^2(G,\mathbb{R}) = \{0\}$.