Math 670: Day 16

Recall from last time:

Corollary: The cohomology of G-invariant forms on G/H is isomorphic to its de Rham cohomology.

Semi-Trivial Ex: S^3 is a Lie group since it's the unit quaternions, so of course $S^3 = S^3/\{1\}$ is a homogeneous space.

Our forms α, β, γ from before are obviously left-invariant (since their dual vector fields X, Y, Z are), & all left-invariant 1-forms are of the form $a\alpha + b\beta + c\gamma$ for constants $a, b, c \in \mathbb{R}$, so the space of left-invariant forms is 3-dimensional.

Likewise, $\alpha\wedge\beta$, $\beta\wedge\gamma$, $\gamma\wedge\alpha$ form a basis for the left-invariant 2-forms. Now, we saw that

$d\alpha = 2\beta\wedge\gamma$, $d\beta = 2\gamma\wedge\alpha$, $d\gamma = 2\alpha\wedge\beta$, so there are no closed left-invariant 1-forms.

In turn, $d(\beta\wedge\gamma) = d\beta\wedge\gamma - \beta\wedge d\gamma = (2\gamma\wedge\alpha)\wedge\gamma - \beta\wedge(2\alpha\wedge\beta) = 0$, etc., so the left-invariant 2-forms are all closed, but they're also all exact.

The left-invariant 0-forms are the constant functions & the left-invariant 3-forms are constant multiples of $\alpha\wedge\beta\wedge\gamma$.

So the cohomology of the complex of left-invariant forms on S^3 looks like:

$$\underset{0}{\mathbb{R}} \qquad \underset{1}{\{0\}} \qquad \underset{2}{\{0\}} \qquad \underset{3}{\mathbb{R}}$$

which indeed agrees w/ the usual cohomology.

Lie Groups

Def: A *Lie group* G is a group which is a differentiable manifold s.t. the map $G\times G \to G$ given by $(g,h) \mapsto gh^{-1}$ is smooth.

Ex: $\mathbb{R}^n$, $S^0 = \mathbb{Z}/2$, $S^1 = U(1)$, $S^3 = SU(2) = Sp(1)$, $\mathbb{RP}^3 = SO(3)$, $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $T^n = (S^1)^n$, the Heisenberg group $\left\{\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1\end{pmatrix}\right\}$, products of Lie groups, etc.

Every Lie group has an associated Lie algebra &, in fact, simply connected Lie groups are completely determined by their Lie algebras, so these are very important beasts.

Def: A real vector space $\mathfrak{g}$ is a **Lie algebra** if it has a bilinear operator $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (the **bracket**) s.t. $\forall\, x,y,z\in\mathfrak{g}$

(i) $[x,x]=0$ (alternating [$\Rightarrow$ anticommutivity])

(ii) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ (Jacobi identity)

Ex: $\mathfrak{X}(M)$ for any M

- $Mat_{n\times n}(\mathbb{R})=\mathfrak{gl}(n,\mathbb{R})$ w/ $[A,B]=AB-BA$.
- $\mathbb{R}^3$ w/ $[\vec{v},\vec{w}]=\vec{v}\times\vec{w}$.

Define left & right translation by $g\in G$ by $L_g(h)=gh$ & $R_g(h)=hg$. A v.f. $X\in\mathfrak{X}(G)$ is **left-invariant** if

$$dL_g X = X \quad \forall\, g\in G.$$

Thm: Suppose G is a Lie group & $\mathfrak{g}$ is the set of left-invt vector fields on G. Then

(i) $\mathfrak{g}$ is a real vector space & the evaluation map $\varphi_e:\mathfrak{g}\to T_eG$ given by $\varphi_e(X)=X_e$ is a v.s. isomorphism,

so $\dim\mathfrak{g}=\dim T_eG=\dim G$.

(ii) Elements of $\mathfrak{g}$ are smooth vector fields.

(iii) For $X,Y\in\mathfrak{g}$, $[X,Y]\in\mathfrak{g}$, where $[\cdot,\cdot]$ is the usual Lie bracket of vector fields

(iv) $(\mathfrak{g},[\cdot,\cdot])$ is a Lie algebra.