Math 670: Day 15

Recall we have the cochain complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M)$$

Define the k^{th} de Rham cohomology group of M to be

$$H^k_{dR}(M) := \ker d_k / \operatorname{im} d_{k-1}$$

Ex: The 1-form $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ on $\mathbb{R}^2 - \{\vec{0}\}$ (or on S^1) is closed (in the kernel of the exterior derivative) but not exact (in the image of the exterior derivative). So it represents a non-trivial element of $H^1_{dR}(\mathbb{R}^2 - \{\vec{0}\})$ (or $H^1_{dR}(S^1)$).

de Rham cohomology is a homotopy invariant of manifolds, meaning homotopy equivalent mfds have identical de Rham cohomology groups.

de Rham Thm: There exists a vector space isomorphism

$$H^k_{dR}(M) \longrightarrow H_k(M;\mathbb{R})^* \simeq H^k(M;\mathbb{R}).$$

In general, computing deRham groups can be hairy... after all, the $\Omega^k(M)$ are $C^\infty(M)$-modules, so they're infinite-dimensional as real vector spaces.

However, when the manifold has lots of symmetry, this becomes a finite-dim'l problem.

Manifolds with the appropriate amount of symmetry are called homogeneous spaces. We will need to build up the tools to deal with them, but here's a sample of the sort of thing that we can do:

A homogeneous space is a manifold admitting a transitive action of a Lie group of diffeomorphisms.

In this case, if M is the homogeneous space & G is the Lie group, let $p \in M$ & let H be the subgroup of G fixing p. Then M turns out to be diffeomorphic to G/H.

Ex: $S^n \cong SO(n+1)/SO(n)$, $\mathbb{CP}^n \cong SU(n+1)/SU(n)$, $\widetilde{G}_k\mathbb{R}^n \cong \frac{SO(n)}{SO(k)\times SO(n-k)}$, etc.

Def: A diff'ble k-form φ on G/H is G-invariant if $l_g^*\varphi = \varphi$ $\forall g \in G$.

Here $l_g : G/H \to G/H$ is given by $l_g(fH) = gfH$, induced by the corresponding map $L_g : G \to G$, $f \mapsto gf$.

Commutative diagram

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ G/H \arrow[r, "l_g"] & G/H \end{tikzcd}$$

Now, if φ is inv't on G/H, then $\pi^*\varphi$ is inv't on G since $L_g^*\pi^*\varphi = (\pi \circ L_g)^*\varphi = (l_g \circ \pi)^* = \pi^* l_g^* \varphi = \pi^*\varphi$.

Thm: Let G be a compact, connected Lie gp, H a Lie subgp, & G/H the corresponding homogeneous space. Then

(i) Each closed k-form on G/H is cohomologous to a G-inv't closed k-form.

(ii) If a G-inv't closed k-form is exact, then it is also the exterior derivative of a G-inv't $(k-1)$-form.

Corollary: The cohomology of G-inv't forms on G/H is isomorphic to its deRham cohomology.
