Math 470: Day 14

Continuing w/ the example from last time:

Ex: Consider S^3 as the unit quaternions. Then at each $p \in S^3$, T_pS^3 can be thought of as consisting of those quaternions which are perpendicular to p.

For example, at $1 \in S^3$, $T_1 S^3 = \{ai + bj + ck\}$.

In general, we can give 3 mutually orthonormal vector fields $X, Y, Z \in \mathfrak{X}(S^3)$ by

$$X_p = ip, \quad Y_p = jp, \quad Z_p = kp \qquad \forall\, p \in S^3.$$

Check that these are tangent vectors: $\langle p, ip\rangle = \langle p_0 + p_1 i + p_2 j + p_3 k, -p_1 + p_0 i - p_3 j + p_2 k\rangle = 0$ etc.

& check that they're mutually perpendicular: $\langle ip, jp\rangle = \langle -p_1 + p_0 i - p_3 j + p_2 k, -p_2 + p_3 i + p_0 j - p_1 k\rangle = 0$, etc.

(In fact, a better argument would be that mult. by $p \in S^3$ is an isometry of $\mathbb{R}^4 = \mathbb{H}$, so orthogonality at 1 carries over)

Now, the corresponding flows are

$$\xi_t(p) = (\cos t + i \sin t)p, \quad \nu_t(p) = (\cos t + j \sin t)p, \quad \zeta_t(p) = (\cos t + k \sin t)p$$

And

$$[X, Y] = \mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (d\xi_{-t})_{\xi_t(p)}\left(Y_{\xi_t(p)}\right).$$

Now, $Y_{\xi_t(p)} = j\,\xi_t(p) = j(\cos t + i \sin t)p = (j \cos t - k \sin t)p$

and $d\xi_{-t} = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}$, so $d\xi_{-t}\left(Y_{\xi_t(p)}\right) = \begin{pmatrix} -p_2 \cos(2t) + p_3 \sin(2t) \\ p_3 \cos(2t) + p_2 \sin(2t) \\ p_0 \cos(2t) - p_1 \sin(2t) \\ -p_1 \cos(2t) - p_0 \sin(2t) \end{pmatrix}$

and therefore $\quad \dfrac{d}{dt}\Big|_{t=0} d\xi_{-t}\left(Y_{\xi_t(p)}\right) = 2\begin{pmatrix} p_3 \\ p_2 \\ -p_1 \\ -p_0 \end{pmatrix} = -2kp = -2Z_p$

Therefore, in general $[X, Y] = -2Z$ &, by symmetry, $[Y, Z] = -2X$, $[Z, X] = -2Y$.

Now, suppose α, β, γ are dual to X, Y, Z, respectively (meaning $\alpha(X) = 1$, $\alpha(Y) = 0$, $\alpha(Z) = 0$, etc.)

Then $d\alpha(Y,Z) = (\iota_Y d\alpha)(Z) \overset{\text{Cartan}}{=} (\mathcal{L}_Y\alpha)(Z) - d(\iota_Y\alpha)(Z) = (\mathcal{L}_Y\alpha)(Z) - d(\cancelto{0}{\alpha(Y)})(Z) = (\mathcal{L}_Y\alpha)(Z)$.

Now, the Lie derivative turns out to be a derivation, so it obeys a Leibniz Rule:

$$\mathcal{L}_u(\omega(Y_1,\dots,Y_k)) = (\mathcal{L}_u\omega)(Y_1,\dots,Y_k) + \sum \omega(Y_1,\dots,\mathcal{L}_u Y_j,\dots,Y_k).$$

In particular, $0 = \mathcal{L}_Y(\alpha(Z)) = (\mathcal{L}_Y\alpha)(Z) + \alpha(\mathcal{L}_Y Z) = (\mathcal{L}_Y\alpha)(Z) + \alpha([Y,Z])$.

Therefore, $d\alpha(Y,Z) = (\mathcal{L}_Y\alpha)(Z) = -\alpha([Y,Z]) = -\alpha(-2X) = 2$

Similar computations show that $d\alpha(X,Y) = 0 = d\alpha(X,Z)$, so $d\alpha = 2\beta\wedge\gamma$.

In particular, notice that this means $\alpha\wedge d\alpha = 2\alpha\wedge\beta\wedge\gamma$ is nowhere zero, so α is a (positive) contact form on S^3.
In fact, this is the standard contact form on S^3, which is usually written in (x_1,y_1,x_2,y_2) coordinates on $\mathbb{R}^4=\mathbb{C}^2$ as

$$\alpha = x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2.$$

Cartan's Magic Formula: If $\omega\in\Omega^k(M)$ & $X\in\mathfrak{X}(M)$, then

$$\mathcal{L}_X\omega = \iota_X d\omega + d\iota_X\omega.$$

Proof: If $\omega = u\in\Omega^0(M)$, we know $\mathcal{L}_X u = X(u)$ and

$$\iota_X du + d\iota_X u = du(X) + 0 = X(u),$$

so the formula holds for 0-forms.

Now consider a differential $\omega = du$. If $Y\in\mathfrak{X}(M)$ then

$$(\mathcal{L}_X du)_p(Y) \overset{\text{defn of } \mathcal{L}_X}{=} \frac{d}{dt}\Big|_{t=0}\varphi_t^*(du_{\varphi_t(p)})(Y) \overset{\text{defn of pullback}}{=} \frac{d}{dt}\Big|_{t=0} du(d\varphi_t Y) \overset{\text{chain rule}}{=} \frac{d}{dt}\Big|_{t=0} d(u\circ\varphi_t)(Y) \overset{\text{defn of differential}}{=} \frac{d}{dt}\Big|_{t=0} Y(u\circ\varphi_t)$$

$$\overset{\text{chain Rule}}{=} Y\left(\frac{d(u\circ\varphi_t)}{dt}\Big|_{t=0}\right) \overset{\text{defn of v.f.}}{=} Y(X(u))$$

OTOH, $\iota_X d(du) = 0$ & $d(\iota_X du)(Y) \overset{\text{defn of contraction}}{=} d(du(X))(Y) \overset{\text{defn of differential}}{=} d(X(u))(Y) \overset{\text{defn of differential}}{=} Y(X(u))$, so the formula holds for differentials.

Now, if $\omega = \sum a_I dx_I$ in local coords,

$$\mathcal{L}_X \omega = \sum \Big((\mathcal{L}_X a_I)\, dx_I + a_I (\mathcal{L}_X dx_I) \Big) = \sum \Big(\big((\iota_X d + d\iota_X)\, a_I \big) dx_I + a_I (\iota_X d + d\iota_X)\, dx_I \Big) = (\iota_X d + d\iota_X) \sum a_I\, dx_I = (\iota_X d + d\iota_X)\,\omega .$$

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