Math 670: Day 13

Lie derivatives of forms & tensors

Prop (Proved in Day 6 notes): Let $X, Y \in \mathfrak{X}(M)$, $p \in M$ & φ_t the local flow of X in a nbhd of p. Then

$$[X,Y]_p = \lim_{t\to 0} \frac{Y_{\varphi_t(p)} - (d\varphi_t)_p(Y)}{t} \quad \text{or, equivalently,} \quad \lim_{t \to 0} \frac{(d\varphi_{-t})_{\varphi_t(p)}(Y) - Y_p}{t}$$

The expression on the RHS is also called the Lie derivative of Y w.r.t. X, denoted $L_X Y$.

Prop: $L_{[u,v]} = [L_u, L_v]$ as operators on $\mathfrak{X}(M)$.

Proof: Let $W \in \mathfrak{X}(M)$. Then

$$L_{[u,v]}W - (L_u L_v W - L_v L_u W) = [[u,v],w] - [u,[v,w]] + [v,[u,w]]$$

$$= [[u,v],w] + [[v,w],u] + [[w,u],v] = 0$$

by the Jacobi identity. ∎

Analogously, we can define the Lie derivative of a form ω w.r.t. a v.f. X as

$$(L_X\omega)_p := \lim_{t\to 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) - \omega_p}{t} = \frac{d}{dt}\Big|_{t=0} \varphi_t^*(\omega_{\varphi_t(p)})$$

More generally, for $\alpha \in \mathcal{T}_r^0(M)$, $\beta \in \mathcal{T}_0^s(M)$,

$$(L_X \alpha\otimes\beta)_p := \frac{d}{dt}\Big((d\varphi_{-t})_{\varphi_t(p)}(\alpha_{\varphi_t(p)}) \otimes (\varphi_t^*)_{\varphi_t(p)}(\beta_{\varphi_t(p)})\Big)$$

Extending linearly gives the Lie derivative on $\mathcal{T}_r^s(M)$.

Lemma/Ex: For $f \in C^\infty(M) = \Omega^0(M)$ & $X \in \mathfrak{X}(M)$, $L_X f = X(f)$.

Proof: $L_X f = \frac{d}{dt}\Big|_{t=0} \varphi_t^* f = \frac{d}{dt}\Big|_{t=0}(f\circ\varphi_t) = X(f)$ by definition. ∎

Cartan's Magic Formula: If $\omega \in \Omega^k(M)$, $X \in \mathfrak{X}(M)$, then $L_X\omega = \iota_X d\omega + d\iota_X\omega$

Here ι_X is the contraction operator (or interior product): for $\omega \in \Omega^k(M)$ & $X \in \mathfrak{X}(M)$, $(\iota_X \omega)_p(Y_1, \dots, Y_{k-1}) = \omega_p(X, Y_1, \dots, Y_{k-1})$

This is a derivation of degree -1.

What does Cartan's Magic Formula get you?

It's one of the most useful tools there is for computing exterior derivatives.

Ex: Consider S^3 as the unit quaternions. Then at each $p \in S^3$, T_pS^3 can be thought of as consisting of the quaternions which are perpendicular to p.

For example, at $1 \in S^3$, $T_1 S^3 = \{ai + bj + ck\}$.

In general, we can give 3 mutually orthonormal vector fields $X, Y, Z \in \mathfrak{X}(S^3)$ by

$$X_p = ip, \quad Y_p = jp, \quad Z_p = kp \quad \forall p \in S^3.$$

Check that these are tangent vectors: $\langle p, ip \rangle = \langle p_0 + p_1 i + p_2 j + p_3 k, -p_1 + p_0 i - p_3 j + p_2 k \rangle = 0$ etc.

& check that they're mutually perpendicular: $\langle ip, jp \rangle = \langle -p_1 + p_0 i - p_3 j + p_2 k, -p_2 + p_3 i + p_0 j - p_1 k \rangle = 0$, etc.

(In fact, a better argument would be that mult. by $p \in S^3$ is an isometry of $\mathbb{R}^4 = \mathbb{H}$, so orthogonality at 1 carries over)