Math 670: Day 12

Def: The exterior derivative $d:\Omega^\cdot(M)\to\Omega^\cdot(M)$ is the unique anti-derivation of degree $+1$ s.t.

① $\forall f\in\Omega^0(M)=C^\infty(M)$, df is the differential of f.

② $d\circ d=0$.

Prop: There is such a beast.

Proof: In local coords, $\omega_p=\sum f_I\,dx_I$, and we define

$$(d\omega)_p=\sum (df_I)\wedge dx_I$$

We saw last time why this is a reasonable way to define the exterior derivative.
Independence of coordinates will follow from uniqueness, so it remains to show:

(i) This is an anti derivation

(ii) It satisfies the cocycle condition ②

(iii) uniqueness.

(i) & (ii) are both computations. Here's the computation for (ii):

If $\omega_p=\sum f_I dx_I$, then

$$(d\omega)_p=\sum_I df_I\wedge dx_I=\sum_I\Big(\sum_i \frac{\partial f}{\partial x_i}dx_i\Big)\wedge dx_I,$$

so $$(dd\omega)_p=\sum_I\Big(\sum_i d\Big(\frac{\partial f}{\partial x_i}\Big)\wedge dx_i\Big)\wedge dx_I=\sum_I\Big(\sum_i\Big(\sum_j\frac{\partial^2 f}{\partial x_j\partial x_i}dx_j\Big)\wedge dx_i\Big)\wedge dx_I=0$$

since $\frac{\partial^2 f}{\partial x_j\partial x_i}=\frac{\partial^2 f}{\partial x_i\partial x_j}$ but $dx_j\wedge dx_i=-dx_i\wedge dx_j$.

For (iii), suppose $\exists$ a degree 1 anti-derivation $D:\Omega^\cdot(M)\to\Omega^\cdot(M)$ satisfying ① & ②. In particular, $Df=df$ $\forall f\in C^\infty(M)$.

Then $D(dx_I)=D(dx_{i_1}\wedge\cdots\wedge dx_{i_k})=\sum_j(-1)^{j-1}dx_{i_1}\wedge\cdots\wedge D\,dx_{i_j}\wedge\cdots\wedge dx_{i_k}=0$ b/c $D(dx_j)\overset{①}{=}D(Dx_j)\overset{②}{=}0$ by hypothesis.

Therefore, if $\omega_p=\sum f_I dx_I$,

$$(D\omega)_p\overset{\text{anti derivation condition}}{=}\sum_I\big(Df_I\wedge dx_I+f_I\,\cancelto{0}{D(dx_I)}\big)\overset{①}{=}\sum df_I\wedge dx_I=(d\omega)_p.$$

This is true for all $p\in M$, so $D\omega = d\omega$. ∎

Ex: If $\omega = f\,dx + g\,dy + h\,dz \in \Omega^1(\mathbb{R}^3)$, then

$$d\omega = \left(\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)\wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial z}dz\right)\wedge dy + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)\wedge dz$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)dz\wedge dx + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)dy\wedge dz.$$

OTOH, $\omega \longleftrightarrow v = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$

& $\nabla\times v = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\frac{\partial}{\partial z} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\frac{\partial}{\partial x} \longleftrightarrow d\omega.$

So d really does generalize the curl.

Also, if $\eta = a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy \longleftrightarrow u = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$, then

$$d\eta = \frac{\partial a}{\partial x}dx\wedge dy\wedge dz + \frac{\partial b}{\partial y}dy\wedge dz\wedge dx + \frac{\partial c}{\partial z}dz\wedge dx\wedge dy = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)dx\wedge dy\wedge dz \longleftrightarrow \nabla\cdot u = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}.$$

So we've succeeded in defining everything in the following diagram & showed that it commutes (here M is a Riemannian 3-mfld):

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \Omega^3(M) \\
C^\infty(M) \arrow[u, equal] \arrow[r, "\nabla"] & \mathfrak{X}(M) \arrow[u, "\flat", "\cong"'] \arrow[r, "\nabla\times"] & \mathfrak{X}(M) \arrow[u, "*\flat", "\cong"'] \arrow[r, "\nabla\cdot"] & C^\infty(M) \arrow[u, "\cong"']
\end{tikzcd}$$

Using HW Problem 2(b), if $f: M^m \to N^n$ is smooth, then we get a map $f^*: \Omega^\cdot(N) \to \Omega^\cdot(M)$ by

$$(f^*\omega)_p(X_1,\dots,X_k) = \omega_{f(p)}(df_pX_1,\dots,df_pX_k)$$

Prop: $d(f^*\omega) = f^*d\omega$ for any $\omega\in\Omega^k(N)$.

Why is this useful? Well, it means you can compute exterior derivatives in more convenient mflds. Later on, we'll consider submersions

$p: SO(n) \to G_k(\mathbb{R}^n)$ or $q: U(n) \to G_k(\mathbb{C}^n)$ ... then we'll be able to compute exterior derivatives up in $SO(n)$ or $U(n)$, where things are nicer.

Proof of Prop: Let $p \in M$. If $\omega = u \in \Omega^0(N)$, then $f^*u = u \circ f$, so $d(f^*u) = d(u \circ f)$. On the other hand,

$$(f^*du)_p(X) = du_{f(p)}(df_p X) = (df_p X)(u) = X(u \circ f) = d(u \circ f)_p(X) \text{ for all } X \in T_pM,\text{ so}$$

$f^*du = d(u \circ f)$, completing the proof in the case $k=0$.

Now, suppose $k>0$ & that $x_1, \dots, x_n$ are local coords in a nbhd of $f(p) \in M$, meaning that $\omega \in \Omega^k(N)$ has the form

$$\omega_{f(p)} = \sum g_I \, dx_I \text{ near } f(p), \text{ so } (d\omega)_{f(p)} = \sum dg_I \wedge dx_I.$$ But then

$$(f^* d\omega)_p \overset{\text{HW 2(a)}}{=} \sum (f^* dg_I) \wedge f^* dx_I \overset{\text{from } k=0 \text{ case}}{=} \sum d(g_I \circ f) \wedge f^* dx_{i_1} \wedge \dots \wedge f^* dx_{i_k} \overset{\text{from } k=0 \text{ case}}{=} \sum d(g_I \circ f) \wedge d(x_{i_1} \circ f) \wedge \dots \wedge d(x_{i_k} \circ f).$$

OTOH, $$(f^*\omega)_p = \sum (g_I \circ f) \, f^* dx_I = \sum (g_I \circ f) \, d(x_{i_1} \circ f) \wedge \dots \wedge d(x_{i_k} \circ f)$$

so $$(d f^* \omega)_p = \sum d(g_I \circ f) \wedge d(x_{i_1} \circ f) \wedge \dots \wedge d(x_{i_k} \circ f).$$ $\blacksquare$