Math 670: Day 11

Remember that a k-form on M^n is a smooth section of $\Lambda^k(M) = \bigcup_{p\in M} \Lambda^k((T_pM)^*)$. The space of k-forms on M is denoted $\Omega^k(M)$ & we let $\Omega^\cdot(M) = \bigoplus_{k\geq 0} \Omega^k(M)$.

Let (U,φ) be a coord. chart in a nbhd of $p\in M$. For each $i=1,\dots,n$, define $x_i : \varphi(U)\to U$ by $x_i(\varphi(a_1,\dots,a_n)) = a_i$.

Then $\{\frac{\partial}{\partial x_i}\}$ is the basis of T_pM associated to φ & $\{dx_i\}$ gives a basis for the dual space $(T_pM)^* = \Lambda^1((T_pM)^*)$, where this is literally just the differential of the function x_i &, by definition of the differential, $dx_{i_p}(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_j}(x_i) = \delta_{ij}$.

Then of course it follows that $\{dx_{i_1}\wedge\dots\wedge dx_{i_k} : 1\leq i_1 < i_2 < \dots < i_k \leq n\}$ gives a basis for $\Lambda^k((T_pM)^*)$ & hence every $\omega\in\Omega^k(M)$ takes the local form $\omega_p = \sum a_I dx_I$ where $I=(i_1,\dots,i_k)$ & $dx_I = dx_{i_1}\wedge\dots\wedge dx_{i_k}$.

Ex: On $\mathbb{R}^2$ we have global coords, so every $\omega\in\Omega^1(\mathbb{R}^2)$ is of the form $\omega = f dx + g dy$ for $f,g\in C^\infty(\mathbb{R}^2)$. So if

$\omega = f dx + g dy$

$\eta = h dx + m dy$

Then $\omega\wedge\eta = (f dx + g dy)\wedge(h dx + m dy) = (fm - gh)\, dx\wedge dy$

so $\omega\wedge\eta(U,V) = \omega\wedge\eta\left(u_1\frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y}, v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\right)$

$= (fm - gh)(u_1v_2 - u_2v_1)$

On $\mathbb{R}^3$, if $\omega_i\in\Omega^1(\mathbb{R}^3)$ w/ $\omega_i = a_i dx + b_i dy + c_i dz$, then

$$\omega_1\wedge\omega_2 = (a_1b_2 - a_2b_1)\, dx\wedge dy + (a_1c_2 - a_2c_1)\, dx\wedge dz + (b_1c_2 - b_2c_1)\, dy\wedge dz$$

Also, if $\omega = a dx + b dy + c dz \in \Omega^1(\mathbb{R}^3)$ & $\eta = f dx\wedge dy + g dx\wedge dz + h dy\wedge dz$, then

$$\omega\wedge\eta = (ah - bg + cf)\, dx\wedge dy\wedge dz$$

Now, to define the exterior derivative, we need to talk about derivations.

Def: An endomorphism A of $\Lambda(V)$ is

- a derivation if $A(u\wedge v) = Au\wedge v + u\wedge Av$

· an anti-derivation if $A(u \wedge v) = Au \wedge v + (-1)^p u \wedge Av$ for $u \in \Lambda^p(V)$ $\left(\Rightarrow A(v_1 \wedge \cdots \wedge v_k) = \sum_i (-1)^{i-1} v_1 \wedge \cdots \wedge Av_i \wedge \cdots \wedge v_k \right)$

· of degree k if $A: \Lambda^i(V) \to \Lambda^{i+k}(V)$ for all i.

This definition extends to any (graded) algebra by swapping in the multiplication on the algebra for $\wedge$ in the above.

Ex: We saw before that tangent vectors can be viewed as (ungraded) derivations on the algebra $C^\infty(M)$. No degree b/c ungraded.

Now, the point is that we can give a universal definition for the exterior derivative.

Def: The exterior derivative $d: \Omega^\cdot(M) \to \Omega^\cdot(M)$ is the unique anti-derivation of degree $+1$ s.t.

① $\forall f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f.

② $d \circ d = 0$.

Prop: There is such a beast.

Idea: In local coords, $\omega_p = \sum f_I \, dx_I$, and we will define

$$(d\omega)_p = \sum (df_I) \wedge dx_I$$

Why? Well, assume there is an operator d satisfying ① & ②. Then $dx_i = d(x_i)$ is d of the 0-form (a.k.a. function) x_i, so $d(dx_i) = (d \circ d) x_i$, which had better be zero if d satisfies the cocycle condition ②.

But then, if d is an anti derivation we will have that $d(dx_I) = 0$ by applying the anti-derivation property iteratively:

$$d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \cancelto{0}{d(dx_{i_1})} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} - dx_{i_1} \wedge \cancelto{0}{d(dx_{i_2})} \wedge \cdots \wedge dx_{i_k} + \cdots + (-1)^{k-1} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge \cancelto{0}{d(dx_{i_k})} = 0.$$

But then

$$(d\omega)_p = d\left(\sum f_I \, dx_I\right) = \sum (df_I) \wedge dx_I + f_I \, \cancelto{0}{d(dx_I)} = \sum (df_I) \wedge dx_I .$$