Math 670: Day 10

A Digression on $\Lambda^k(V)$ & Grassmannians

Let $\widetilde{G}_k(\mathbb{R}^n)$ be the Grassmannian of oriented k-planes in $\mathbb{R}^n$. e.g., $\widetilde{G}_1(\mathbb{R}^n) \cong S^{n-1}$, $\widetilde{G}_2\mathbb{R}^3$ = oriented planes in $\mathbb{R}^3 \cong \widetilde{G}_1\mathbb{R}^3 \cong S^2$, etc.

Now, we can define the Plücker embedding $\Psi : \widetilde{G}_k\mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)/\mathbb{R}_+$ as follows:

For $P \in \widetilde{G}_k\mathbb{R}^n$, let $v_1, \ldots, v_k$ be a $\overset{\text{oriented}}{\text{basis}}$ for P & define $\Psi(P) = v_1 \wedge v_2 \wedge \cdots \wedge v_k$.

This is well-defined b/c, if $w_1, \ldots, w_k$ is another $\overset{\text{oriented}}{\text{basis}}$ for P & $A = (a_{ij})$ is the change-of-basis matrix, then we have $w_i = \sum a_{ij} v_j$, so $w_1 \wedge \cdots \wedge w_k = (\sum a_{1j} v_j) \wedge \cdots \wedge (\sum a_{nj} v_j) = (\det A)\, v_1 \wedge \cdots \wedge v_k$, which is equivalent in $\Lambda^k(\mathbb{R}^n)/\mathbb{R}_+$ if $\det A > 0$, as it must be if the two bases determine the same orientation.

(Note: if we want the usual Grassmannian of unoriented k-planes, the corresponding map is to $\Lambda^k(\mathbb{R}^n)/\mathbb{R}^* \ldots$ a.k.a., the projective space $\mathbb{P}(\Lambda^k(\mathbb{R}^n)) \cong \mathbb{RP}^{\binom{n}{k}-1}$. I'm not doing this just b/c it will make a later computation easier)

Now, notice that the image of Ψ is precisely the set of non-zero decomposable elements, meaning those elements of $\Lambda^k(\mathbb{R}^n)/\mathbb{R}_+$ which appear as residue classes of some simple element $v_1 \wedge \cdots \wedge v_k$.

It's easy to show this map is an embedding, so the image of Ψ is a (diffeomorphic) copy of $\widetilde{G}_k(\mathbb{R}^n)$, and hence we can recover a copy by identifying the collection of decomposable elements of $\Lambda^k(\mathbb{R}^n)$.

Let's specialize to $\widetilde{G}_2(\mathbb{R}^4)$, which is the first interesting example.

We just said, we can identify $\widetilde{G}_2(\mathbb{R}^4)$ w/ the decomposable elements of $\Lambda^2(\mathbb{R}^4)$ mod scaling. So rescale to length 1 & we're talking about some subset of the unit sphere in $\Lambda^2(\mathbb{R}^4) \cong \mathbb{R}^6$.

If $e_1, \ldots, e_4$ is an ON basis for $\mathbb{R}^4$, then (HW problem) $e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4$ is an ON basis for $\Lambda^2(\mathbb{R}^4)$

So what are the unit length, decomposable vectors in $\Lambda^2(\mathbb{R}^4)$?

Prop: $\omega \in \Lambda^2\mathbb{R}^4$ is decomposable $\iff$ $\omega \wedge \omega = 0$.

Proof: Exercise. $\blacksquare$

Now, as in the HW, define the involution $\star$ on $\Lambda^2\mathbb{R}^4$ by

$$e_1\wedge e_2 \overset{\star}{\longleftrightarrow} e_3\wedge e_4$$
$$e_1\wedge e_3 \overset{\star}{\longleftrightarrow} -e_2\wedge e_4 \quad \text{\& extend linearly.}$$
$$e_1\wedge e_4 \overset{\star}{\longleftrightarrow} e_2\wedge e_3$$

Note: $\star\Psi(P) = \Psi(P^\perp)$ where $P^\perp$ is the orthogonal complement of P.

Prop: $\omega \in \Lambda^2\mathbb{R}^4$ is decomposable $\iff$ ω is perpendicular to $\star\omega$.

Proof: If $\omega = \sum a_{ij}\, e_i \wedge e_j$, then $\langle \omega, \star\omega\rangle = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$.

OTOH, $\omega\wedge\omega = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\, e_1\wedge e_2\wedge e_3\wedge e_4$, so $\langle \omega, \star\omega\rangle = 0 \iff \omega\wedge\omega = 0 \iff \omega$ decomposable. $\blacksquare$

Now, $\star$ is an isometric (distance-preserving) involution, so the only possible eigenvalues are ± 1.

Let E_+ be the $+1$-eigenspace & let E_- be the -1-eigenspace.

Note: E_+ & E_- are perpendicular since $\star$ is symmetric, which you can see since the matrix of $\star$ is
$$\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&-1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&-1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}.$$

Then for any $\omega \in \Lambda^2(\mathbb{R}^4)$, $\omega = \frac{\omega+\star\omega}{2} + \frac{\omega-\star\omega}{2}$ with $\frac{\omega+\star\omega}{2} \in E_+$ & $\frac{\omega-\star\omega}{2} \in E_-$.

So then it follows that $\frac{e_1\wedge e_2 \pm e_3\wedge e_4}{2}$, $\frac{e_1\wedge e_3 \mp e_2\wedge e_4}{2}$, $\frac{e_1\wedge e_4 \pm e_2\wedge e_3}{2}$ are bases for $E_\pm$.

Prop: $\omega \in \Lambda^2\mathbb{R}^4$ is decomposable $\iff$ $\frac{\omega+\star\omega}{2}$ & $\frac{\omega-\star\omega}{2}$ are the same length.

Proof: ω decomposable $\iff 0 = \langle \omega, \star\omega\rangle = \left\langle \frac{\omega+\star\omega}{2} + \frac{\omega-\star\omega}{2}, \frac{\omega+\star\omega}{2} - \frac{\omega-\star\omega}{2}\right\rangle = \left\|\frac{\omega+\star\omega}{2}\right\|^2 - \left\|\frac{\omega-\star\omega}{2}\right\|^2$. $\blacksquare$

Cor: $\omega \in \Lambda^2\mathbb{R}^4$ is unit length & decomposable $\iff$ $\frac{\omega+\star\omega}{2}$ & $\frac{\omega-\star\omega}{2}$ both have norm $\frac{1}{\sqrt{2}}$.

Therefore, if $S^2_+(1/\sqrt{2})$ is the round sphere of radius $\frac{1}{\sqrt{2}}$ in E_+ & $S^2_-(1/\sqrt{2})$ is the round sphere of radius $\frac{1}{\sqrt{2}}$ in E_-,

then we can identify $\tilde{G}_2\mathbb{R}^4$ with $S^2_+(1/\sqrt{2}) \times S^2_-(1/\sqrt{2})$ (in fact, this identification turns out to be an isometry.)

Concretely, if $(p,q) \in S^2_+(1/\sqrt{2}) \times S^2_-(1/\sqrt{2})$, then the corresponding 2-plane in $\mathbb{R}^4$ is given by the following:

$$p=(p_1,p_2,p_3) \longleftrightarrow p_1\frac{e_1\wedge e_2+e_3\wedge e_4}{2}+p_2\frac{e_1\wedge e_3-e_2\wedge e_4}{2}+p_3\frac{e_1\wedge e_4+e_2\wedge e_3}{2}=\frac{\omega+*\omega}{2}$$

$$q=(q_1,q_2,q_3) \longleftrightarrow q_1\frac{e_1\wedge e_2-e_3\wedge e_4}{2}+q_2\frac{e_1\wedge e_3+e_2\wedge e_4}{2}+q_3\frac{e_1\wedge e_4-e_2\wedge e_3}{2}=\frac{\omega-*\omega}{2},$$ so

$$\omega=\frac{\omega+*\omega}{2}+\frac{\omega-*\omega}{2}=(p_1+q_1)e_1\wedge e_2+(p_2+q_2)e_1\wedge e_3+(p_3+q_3)e_1\wedge e_4+(p_3-q_3)e_2\wedge e_3+(-p_2+q_2)e_2\wedge e_4+(p_1-q_1)e_3\wedge e_4,$$

(Note that $(p_1+q_1)(p_1-q_1)-(p_2+q_2)(-p_2+q_2)+(p_3+q_3)(p_3-q_3)=p_1^2-q_1^2+p_2^2-q_2^2+p_3^2-q_3^2=\|p\|^2-\|q\|^2=\frac{1}{2}-\frac{1}{2}=0$,

so this really does satisfy the Plücker relation $a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}=0$)

Then $\omega=\big((p_3+q_3)e_1-(p_2-q_2)e_2+(p_1-q_1)e_3\big)\wedge\big((p_1+q_1)e_2+(p_2+q_2)e_3+(p_2+q_2)e_4\big)$

corresponds to the plane spanned by $(p_3+q_3,\ q_2-p_2,\ p_1-q_1,\ 0)$ & $(0,\ p_1+q_1,\ p_2+q_2,\ p_2+q_2)$ in $\mathbb{R}^4$.